The Statistical Limit of Arbitrage

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Rapid advances in machine learning should, in theory, make statistical arbitrage investing a lot easier and more lucrative.

See a recent survey: “Financial Machine Learning”, Kelly and Xiu (2023, FnT Finance)

Is there a ceiling for gains in machine-learned arbitrage?

Yes! And this ceiling is much lower than what investors might anticipate

Pricing errors (alphas) ≠ Arbitrage opportunities
Ross (1976)'s arbitrage pricing theory (APT) rules out near-arbitrage opportunities based on true parameters in the return generating process.

- Implicit assumption: arbitrageurs know true parameters, as in rational expectations asset pricing models following tradition of Lucas (1978).

- Yet, arbitrageurs in practice learn about profit opportunities from statistical analysis prior to investment.

- The challenge of learning about a large number of alphas induces a limit to arbitrage due to statistical learning.

- Statistical limits to arbitrage is absent from existing theories of limits to arbitrage.
Earning a tiny spread on each of thousands of trades, as if it were vacuuming up nickels that others couldn't see.
Challenge Facing Arbitrageurs: Real or Fake?
Summary of Theoretical Results

- We document a statistical limit to arbitrage in a linear asset pricing model where arbitrageurs are only allowed to employ a feasible trading strategy that relies on historical data to make inference on alpha signals.

- We derive the optimal Sharpe ratio achievable by any feasible arbitrage trading strategies.

- We shed light on a theoretical gap between the infeasible Sharpe ratio (pricing errors) and the feasible Sharpe ratio (investment opportunities).

- We show how arbitrageurs can design an “all-weather” feasible strategy that achieves the optimality.

- We examine alternative machine learning strategies that exploit multiple testing, shrinkage, and selection techniques.
Summary of Empirical Findings

- The cross-sectional $R^2$s of a 27-factor model we built akin to BARRA are rather low, with a time-series average 8.25% from Jan 1965 to Dec 2020.

- Among 12,415 test statistics in total, only 6.35% (0.63%) of the t-statistics are greater than 2.0 (3.0) in absolute values.

- Feasible arbitrage portfolios achieve a moderately low annualized Sharpe ratio, about 0.5, whereas the perceived (infeasible) Sharpe ratios are considerably higher, around 3.0.
Return Model

- $N$ assets in the market. Their excess returns follow a linear factor model.

- For the talk: An $N$-dimensional return vector follows:

$$r_t = \alpha + u_t, \quad \text{where} \quad u_{i,t} \sim \mathcal{N}(0,1),$$

where alphas are i.i.d. across assets.
Classical Arbitrage Pricing Theory

▶ With perfect knowledge of $\alpha$, the mean-variance optimal arbitrage portfolio is

$$w = \alpha,$$

which achieves a Sharpe ratio

$$S^* := \sqrt{\alpha^\top \alpha}.$$

▶ Ingersoll (1984, JF) shows that the condition for absence of near-arbitrage:

$$S^* \leq C,$$

for some constant $C > 0$.

▶ But near-arbitrage under knowledge of $\alpha \neq$ near-arbitrage w/o knowledge of $\alpha$
But arbs do not observe true DGP. They infer $\alpha$ from a sample of size $T$:

$$\hat{\alpha} = \alpha + \bar{u}, \quad \bar{u} \sim \mathcal{N}(0, 1/T).$$

Suppose further they form an arbitrage portfolio with weights

$$\hat{\omega} = \hat{\alpha}.$$
Statistical Limits: Sharpe Ratio Gap

- OOS Sharpe ratio $S$ achieved by $\hat{\omega} = \hat{\alpha}$ vs. $S^*$ based on $w = \alpha$:
  \[
  \frac{(S^*)^2}{S^2} = 1 + \frac{1}{T} + \frac{N}{T(S^*)^2}.
  \]

- The gap can be substantial when alphas sufficiently rare and/or small!
Statistical Limits: A Non-trivial Example

▶ Suppose alphas are drawn from:

\[ \alpha_i \sim \begin{cases} 
\mu & \text{with prob. } \frac{\rho}{2} \\
-\mu & \text{with prob. } \frac{\rho}{2} \\
0 & \text{with prob. } 1 - \rho
\end{cases}, \quad 1 \leq i \leq N. \]

▶ Suppose only a small portion of assets have a nonzero yet small alpha:

\[ \mu \sim T^{-1/2}, \quad \rho \sim N^{-1/2} \]

▶ Suppose \( N^{1/2} / T \to \infty \) and \( T \to \infty \).

▶ Infeasible strategy \( w = \alpha \) generates

\[ S^* \sim N^{1/2} / T \to \infty, \]

▶ Feasible strategy \( w = \hat{\alpha} \) generates

\[ S \sim 1 / T \to 0. \]
More Difficult Questions

- Is it possible that a smart arbitrageur can find a strategy that closes this gap?
- What is the optimal feasible strategy?
Impact of Feasibility Constraint

Let $S(\hat{w})$ be the Sharpe ratio generated by any $\hat{w}$.

Feasible strategy: function of historical data (returns from $t - T + 1$ to $t$).

Theorem 1: $\hat{w}$ is feasible $\implies S(\hat{w}) \leq S^{\text{OPT}} + o_p(1)$, $(S^{\text{OPT}})^2 := E(\alpha|\mathcal{G})^\top E(\alpha|\mathcal{G})$.

Here $\mathcal{G}$ is the information set generated by historical returns.

Not surprisingly, feasible Sharpe ratio smaller than infeasible one:

$$E((S^*)^2) \geq (S^{\text{OPT}})^2.$$  

This provides a solution to the classical problem of optimal portfolio allocation when parameters (mean and covariances) are unknown.
The image contains a table and a graph. The table likely represents a data matrix, and the graph might be a contour plot or a color map. The table has rows and columns labeled with values, and the graph has a color scale indicating different levels or categories. The exact data values and specific context of the table and graph are not clear from the image alone.
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The ratio $S_{\text{OPT}} / S^*$ is shown in the table, where $S_{\text{OPT}}$ represents the optimal solution and $S^*$ represents the solution obtained under the standardization $\mu / \sigma \times \sqrt{12}$. The values in the table indicate the ratio of these two solutions at different values of $\rho \times 100$ and $\mu / \sigma \times \sqrt{12}$.
The optimal feasible strategy is simply $w^{\text{OPT}} = E(\alpha|\mathcal{G})$, which has closed-form given Bayes' rule, if arbitrageurs know the distribution of alphas.

In practice we do not know the distribution.

In this case a feasible strategy $\hat{w}^{\text{OPT}}$ can be constructed using Empirical Bayes via Tweedie's formula:

$$ \hat{w}_i^{\text{OPT}} = \hat{\alpha}_i + \frac{1}{T} \left. \frac{d \log \hat{p}(a)}{da} \right|_{a=\hat{\alpha}_i} $$

Bayes Shrinkage

where $\hat{p}(a)$ is a nonparametric estimator of the marginal density of $\hat{\alpha}_i$.

We show the strategy $\hat{w}^{\text{OPT}}$ achieves the optimal feasible Sharpe $S^{\text{OPT}}$ under (almost) arbitrary alpha distributions.
Strategy: $\hat{w} = \hat{\alpha}$. 
False Discovery Rate Control Strategy: applying Benjamini-Hochberg algo onto $\hat{\alpha}_i$s.
Lasso Strategy: applying Lasso algo onto $\hat{\alpha}_i$s.
Empirical Analysis of US Equities

- We study US monthly equity returns from January 1965 to December 2020.

- We adopt a multi-factor model with 16 characteristics and 11 GICS sectors, including market beta, size, operating profits/book equity, book equity/market equity, asset growth, momentum, short-term reversal, industry momentum, illiquidity, leverage, return seasonality, sales growth, accruals, dividend yield, tangibility, and idiosyncratic risk.

- 10-year rolling window estimation, last 2 years as validation sample for tuning parameter selection.
Time-series of the Cross-sectional $R^2$s

Rare and Weak Alphas

6.35% (0.63%) of the t-Stats > 2.0 (3.0); 0.505% alphas with a Sharpe ratio > 1.0.
Performance of Risk-Normalized Arbitrage Portfolios

Sharpe Ratios: OPT (red, 0.496), CSR (blue, 0.450), BH (green, 0.497), and LASSO (orange, 0.384)

In contrast, average of $\hat{S}^*$ is about 2.95, which is far greater than feasible Sharpe ratios, $\sim 0.5$. 
Average of $\hat{S}^*$ is about 2.95, which is far greater than feasible Sharpe ratios, $\sim 0.5$. 
Conclusion

▶ Statistical limit to arbitrage: Widens the bounds in which mispricing can survive in presence of arbitrageurs.

▶ Existing empirical evidence provides “lower bound” of Sharpe ratios achievable with machine learning methods (based on ad-hoc choices). Our theoretical analysis provides an “upper bound” in a specific context (based on optimal strategy).

▶ The gap between feasible and infeasible Sharpe ratios will further increase if arbitrageurs face additional statistical challenges, e.g., model misspecification, omitted factors, weak factors, large non-sparse covariance matrix.