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The Statistical Limit of Arbitrage

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Introduction

- Rapid advances in machine learning should, in theory, make statistical arbitrage investing a lot easier and more lucrative.
  - See a recent survey: "Financial Machine Learning", Kelly and Xiu (2023, FnT Finance)
- Is there a ceiling for gains in machine-learned arbitrage?
  - Yes! And this ceiling is much lower than what investors might anticipate
  - Pricing errors (alphas) $\neq$ Arbitrage opportunities

Introduction

- Ross (1976)'s arbitrage pricing theory (APT) rules out near-arbitrage opportunities based on *true* parameters in the return generating process
  - Implicit assumption: arbitrageurs know true parameters, as in rational expectations asset pricing models following tradition of Lucas (1978)
- Yet, arbitrageurs in practice learn about profit opportunities from *statistical analysis* prior to investment.
- The challenge of learning about *a large number of* alphas induces a limit to arbitrage due to statistical learning.
- Statistical limits to arbitrage is absent from existing theories of limits to arbitrage.

Myron Scholes' Description of LTCM's Trading Strategy

— Illustration created by Bing powered by OpenAI's Dall-E technology

Challenge Facing Arbitrageurs: Real or Fake?

Summary of Theoretical Results

- We document a statistical limit to arbitrage in a linear asset pricing model where arbitrageurs are only allowed to employ a *feasible* trading strategy that relies on historical data to make inference on alpha signals.
- We derive the optimal Sharpe ratio achievable by any feasible arbitrage trading strategies.
- We shed light on a theoretical gap between the infeasible Sharpe ratio (pricing errors) and the feasible Sharpe ratio (investment opportunities).
- We show how arbitrageurs can design an "all-weather" feasible strategy that achieves the optimality.
- We examine alternative machine learning strategies that exploit multiple testing, shrinkage, and selection techniques.

Summary of Empirical Findings

- The cross-sectional R^2s of a 27-factor model we built akin to BARRA are rather low, with a time-series average 8.25% from Jan 1965 to Dec 2020.
- Among 12,415 test statistics in total, only 6.35% (0.63%) of the t-statistics are greater than 2.0 (3.0) in absolute values.
- Feasible arbitrage portfolios achieve a moderately low annualized Sharpe ratio, about 0.5, whereas the perceived (infeasible) Sharpe ratios are considerably higher, around 3.0.

Return Model

- N assets in the market. Their excess returns follow a linear factor model.
- For the talk: An N-dimensional return vector follows:

$$r_t = \alpha + u_t, \quad \text{where} \quad u_{i,t} \sim \mathcal{N}(0,1),$$

where alphas are i.i.d. across assets.

Classical Arbitrage Pricing Theory

- With perfect knowledge of α, the mean-variance optimal arbitrage portfolio is

$$w = \alpha,$$

which achieves a Sharpe ratio

$$S^\star := \sqrt{\alpha^\intercal \alpha}.$$

- Ingersoll (1984, JF) shows that the condition for absence of near-arbitrage:

$$S^\star \leq C, \quad \text{for some constant } C > 0.$$

- But near-arbitrage under knowledge of $\alpha \neq$ near-arbitrage w/o knowledge of α

A Simple Feasible Strategy

- But arbs do not observe true DGP. They infer α from a sample of size T:

$$\widehat{\alpha} = \alpha + \bar{u}, \quad \bar{u} \sim \mathcal{N}(0, 1/T).$$

- Suppose further they form an arbitrage portfolio with weights

$$\widehat{\omega} = \widehat{\alpha}.$$

Statistical Limits: Sharpe Ratio Gap

- OOS Sharpe ratio S achieved by $\widehat{\omega} = \widehat{\alpha}$ vs. $S^\star$ based on $w = \alpha$:

$$\frac{(S^\star)^2}{S^2} = 1 + \frac{1}{T} + \frac{N}{T(S^\star)^2}.$$

- The gap can be substantial when alphas sufficiently rare and/or small!

Statistical Limits: A Non-trivial Example

- Suppose alphas are drawn from:

$$\alpha_i \overset{i.i.d.}{\sim} \begin{cases} \mu & \text{with prob. } \rho/2 \\ -\mu & \text{with prob. } \rho/2 \\ 0 & \text{with prob. } 1-\rho \end{cases}, \quad 1 \le i \le N.$$

- Suppose only a small portion of assets have a nonzero yet small alpha:

$$\mu \sim T^{-1/2}, \qquad \rho \sim N^{-1/2}$$

- Suppose $N^{1/2}/T \to \infty$ and $T \to \infty$.
- Infeasible strategy $w = \alpha$ generates

$$S^{\star} \sim N^{1/2}/T \to \infty,$$

- Feasible strategy $w = \widehat{\alpha}$ generates

$$S \sim 1/T \to 0.$$

More Difficult Questions

- Is it possible that a smart arbitrageur can find a strategy that closes this gap?
- What is the optimal feasible strategy?

Impact of Feasibility Constraint

- Let $S(\widehat{w})$ be the Sharpe ratio generated by any $\widehat{w}$.
- Feasible strategy: function of historical data (returns from $t-T+1$ to t).
- Theorem 1: $\widehat{w}$ is feasible $\Longrightarrow$

$$S(\widehat{w}) \leq S^{\mathrm{OPT}} + o_{\mathrm{P}}(1), \quad (S^{\mathrm{OPT}})^2 := \mathrm{E}\big(\mathrm{E}(\alpha|\mathscr{G})^{\intercal}\mathrm{E}(\alpha|\mathscr{G})\big).$$

  Here $\mathscr{G}$ is the information set generated by historical returns.
- Not surprisingly, feasible Sharpe ratio smaller than infeasible one:

$$\mathrm{E}((S^{\star})^2) \geq \left(S^{\mathrm{OPT}}\right)^2.$$

- This provides a solution to the classical problem of optimal portfolio allocation when parameters (mean and covariances) are unknown.

S^{OPT}

$\rho \times 100$ \ $\mu/\sigma \times \sqrt{12}$	10.95	8.22	5.48	4.38	3.29	2.74	2.19	1.92	1.64	1.37	1.10	0.99	0.88	0.77	0.66	0.55	0.44	0.33	0.22	0.16	0.11
50.00	70.71	53.03	35.35	28.21	20.83	16.95	12.96	10.94	8.94	6.97	5.04	4.29	3.55	2.85	2.19	1.58	1.05	0.61	0.28	0.16	0.07
25.00	50.00	37.50	24.99	19.92	14.55	11.66	8.64	7.12	5.64	4.22	2.92	2.43	1.98	1.56	1.17	0.83	0.54	0.31	0.14	0.08	0.04
20.00	44.72	33.54	22.35	17.80	12.96	10.33	7.58	6.20	4.85	3.59	2.44	2.02	1.63	1.28	0.96	0.68	0.44	0.25	0.11	0.06	0.03
15.00	38.73	29.05	19.36	15.40	11.16	8.84	6.40	5.17	4.00	2.90	1.93	1.58	1.27	0.98	0.73	0.51	0.33	0.19	0.08	0.05	0.02
10.00	31.62	23.72	15.80	12.56	9.04	7.08	5.02	4.00	3.03	2.14	1.38	1.12	0.88	0.68	0.50	0.35	0.22	0.13	0.06	0.03	0.01
7.50	27.39	20.54	13.68	10.87	7.78	6.04	4.22	3.32	2.47	1.71	1.08	0.87	0.68	0.52	0.38	0.26	0.17	0.09	0.04	0.02	0.01
5.00	22.36	16.77	11.17	8.86	6.29	4.83	3.30	2.55	1.85	1.25	0.76	0.60	0.47	0.35	0.26	0.18	0.11	0.06	0.03	0.02	0.01
2.50	15.81	11.86	7.90	6.24	4.36	3.27	2.14	1.60	1.11	0.71	0.41	0.32	0.24	0.18	0.13	0.09	0.06	0.03	0.01	0.01	0.00
1.25	11.18	8.39	5.58	4.40	3.01	2.20	1.37	0.99	0.66	0.40	0.22	0.17	0.12	0.09	0.07	0.04	0.03	0.02	0.01	0.00	0.00
1.00	10.00	7.50	4.99	3.93	2.67	1.93	1.18	0.84	0.55	0.33	0.18	0.13	0.10	0.07	0.05	0.04	0.02	0.01	0.01	0.00	0.00
0.75	8.66	6.50	4.32	3.39	2.28	1.63	0.98	0.68	0.44	0.25	0.13	0.10	0.08	0.06	0.04	0.03	0.02	0.01	0.00	0.00	0.00
0.50	7.07	5.30	3.53	2.76	1.83	1.28	0.74	0.51	0.31	0.18	0.09	0.07	0.05	0.04	0.03	0.02	0.01	0.01	0.00	0.00	0.00
0.25	5.00	3.75	2.49	1.94	1.25	0.84	0.46	0.30	0.18	0.09	0.05	0.03	0.03	0.02	0.01	0.01	0.01	0.00	0.00	0.00	0.00
0.12	3.54	2.65	1.76	1.36	0.85	0.55	0.28	0.17	0.10	0.05	0.02	0.02	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00

Ratio $S^{\mathrm{OPT}}/S^{\star}$

$\rho \times 100$ \ $\mu/\sigma \times \sqrt{12}$	10.95	8.22	5.48	4.38	3.29	2.74	2.19	1.92	1.64	1.37	1.10	0.99	0.88	0.77	0.66	0.55	0.44	0.33	0.22	0.16	0.11
50.00	1.00	1.00	1.00	1.00	0.98	0.96	0.92	0.88	0.84	0.79	0.71	0.67	0.63	0.58	0.52	0.45	0.37	0.29	0.20	0.15	0.10
25.00	1.00	1.00	1.00	1.00	0.97	0.93	0.86	0.81	0.75	0.68	0.58	0.54	0.49	0.44	0.39	0.33	0.27	0.21	0.14	0.11	0.07
20.00	1.00	1.00	1.00	1.00	0.97	0.92	0.85	0.79	0.72	0.64	0.54	0.50	0.46	0.41	0.36	0.30	0.25	0.19	0.13	0.09	0.06
15.00	1.00	1.00	1.00	0.99	0.96	0.91	0.83	0.76	0.69	0.60	0.50	0.45	0.41	0.36	0.31	0.27	0.21	0.16	0.11	0.08	0.05
10.00	1.00	1.00	1.00	0.99	0.95	0.90	0.79	0.72	0.64	0.54	0.44	0.39	0.35	0.31	0.26	0.22	0.18	0.13	0.09	0.07	0.04
7.50	1.00	1.00	1.00	0.99	0.95	0.88	0.77	0.69	0.60	0.50	0.39	0.35	0.31	0.27	0.23	0.19	0.15	0.12	0.08	0.06	0.04
5.00	1.00	1.00	1.00	0.99	0.94	0.86	0.74	0.65	0.55	0.45	0.34	0.30	0.26	0.23	0.19	0.16	0.13	0.09	0.06	0.05	0.03
2.50	1.00	1.00	1.00	0.99	0.92	0.83	0.68	0.58	0.47	0.36	0.26	0.22	0.19	0.16	0.14	0.11	0.09	0.07	0.04	0.03	0.02
1.25	1.00	1.00	1.00	0.98	0.90	0.79	0.61	0.50	0.39	0.28	0.19	0.16	0.14	0.12	0.10	0.08	0.06	0.05	0.03	0.02	0.02
1.00	1.00	1.00	1.00	0.98	0.89	0.77	0.59	0.48	0.37	0.26	0.18	0.15	0.13	0.11	0.09	0.07	0.06	0.04	0.03	0.02	0.01
0.75	1.00	1.00	1.00	0.98	0.88	0.75	0.56	0.45	0.34	0.23	0.15	0.13	0.11	0.09	0.08	0.06	0.05	0.04	0.02	0.02	0.01
0.50	1.00	1.00	1.00	0.98	0.86	0.73	0.52	0.41	0.30	0.20	0.13	0.11	0.09	0.08	0.06	0.05	0.04	0.03	0.02	0.02	0.01
0.25	1.00	1.00	1.00	0.97	0.83	0.68	0.46	0.34	0.24	0.15	0.09	0.08	0.06	0.05	0.04	0.04	0.03	0.02	0.01	0.01	0.01
0.12	1.00	1.00	1.00	0.96	0.80	0.62	0.39	0.28	0.18	0.11	0.07	0.05	0.05	0.04	0.03	0.03	0.02	0.02	0.01	0.01	0.01

Optimal Feasible Strategy

- The optimal feasible strategy is simply $w^{\mathrm{OPT}} = \mathrm{E}(\alpha|\mathscr{G})$, which has closed-form given Bayes' rule, if arbitrageurs know the distribution of alphas.
- In practice we do not know the distribution.
- In this case a feasible strategy $\widehat{w}^{\mathrm{OPT}}$ can be constructed using Empirical Bayes via Tweedie's formula:

$$\widehat{w}_i^{\mathrm{OPT}} = \widehat{\alpha}_i + \underbrace{\frac{1}{T}\frac{d\log\widehat{p}(a)}{da}\bigg|_{a=\widehat{\alpha}_i}}_{\text{Bayes Shrinkage}},$$

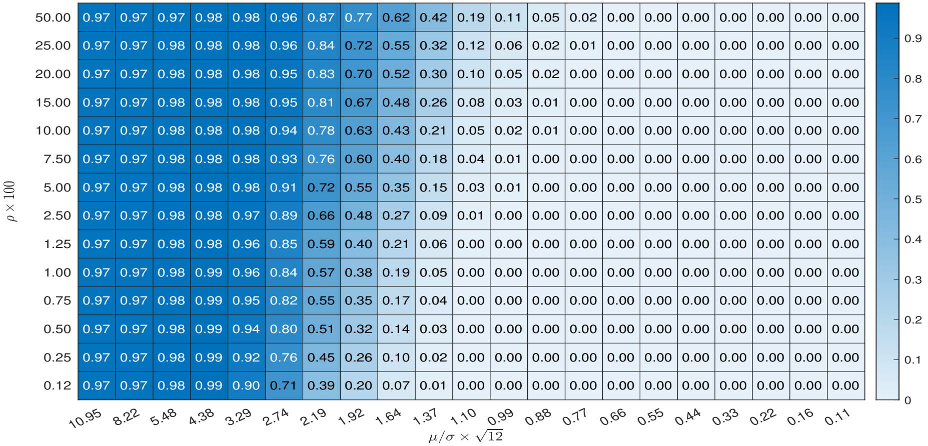
  where $\widehat{p}(a)$ is a nonparametric estimator of the marginal density of $\widehat{\alpha}_i$.
- We show the strategy $\widehat{w}^{\mathrm{OPT}}$ achieves the optimal feasible Sharpe S^{OPT} under (almost) arbitrary alpha distributions.

Ratio $S^{\mathrm{CSR}} / S^{\mathrm{OPT}}$

$\rho \times 100$ \ $\mu/\sigma \times \sqrt{12}$	10.95	8.22	5.48	4.38	3.29	2.74	2.19	1.92	1.64	1.37	1.10	0.99	0.88	0.77	0.66	0.55	0.44	0.33	0.22	0.16	0.11
50.00	1.00	0.99	0.98	0.97	0.97	0.97	0.98	0.98	0.99	0.99	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
25.00	0.99	0.98	0.96	0.95	0.93	0.93	0.95	0.96	0.97	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
20.00	0.99	0.98	0.95	0.93	0.92	0.91	0.93	0.94	0.95	0.97	0.98	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
15.00	0.98	0.97	0.94	0.91	0.89	0.88	0.89	0.91	0.92	0.94	0.96	0.97	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00
10.00	0.98	0.96	0.91	0.88	0.84	0.83	0.84	0.85	0.87	0.90	0.94	0.95	0.96	0.98	0.98	0.99	1.00	1.00	1.00	1.00	1.00
7.50	0.97	0.95	0.89	0.85	0.80	0.79	0.79	0.81	0.83	0.87	0.92	0.93	0.95	0.97	0.98	0.99	1.00	1.00	1.00	1.00	1.00
5.00	0.95	0.92	0.85	0.79	0.73	0.72	0.73	0.74	0.78	0.82	0.89	0.91	0.94	0.96	0.97	0.99	0.99	1.00	1.00	1.00	1.00
2.50	0.91	0.86	0.75	0.68	0.61	0.59	0.60	0.63	0.68	0.75	0.84	0.88	0.91	0.94	0.97	0.98	0.99	1.00	1.00	1.00	1.00
1.25	0.85	0.76	0.62	0.54	0.48	0.47	0.49	0.53	0.59	0.68	0.80	0.85	0.90	0.94	0.96	0.98	0.99	1.00	1.00	1.00	1.00
1.00	0.82	0.73	0.58	0.50	0.44	0.43	0.46	0.50	0.57	0.67	0.80	0.85	0.90	0.93	0.96	0.98	0.99	1.00	1.00	1.00	1.00
0.75	0.77	0.68	0.52	0.45	0.39	0.39	0.42	0.46	0.54	0.65	0.79	0.84	0.89	0.93	0.96	0.98	0.99	1.00	1.00	1.00	1.00
0.50	0.71	0.60	0.45	0.38	0.33	0.33	0.37	0.42	0.50	0.62	0.77	0.83	0.89	0.93	0.96	0.98	0.99	1.00	1.00	1.00	1.00
0.25	0.58	0.47	0.33	0.28	0.25	0.26	0.31	0.36	0.45	0.59	0.76	0.83	0.88	0.93	0.96	0.98	0.99	1.00	1.00	1.00	1.00
0.12	0.45	0.35	0.24	0.20	0.19	0.20	0.25	0.31	0.41	0.56	0.75	0.82	0.88	0.93	0.96	0.98	0.99	1.00	1.00	1.00	1.00

- Strategy: $\widehat{w} = \widehat{\alpha}$.

Ratio $S^{\text{BH}}/S^{\text{OPT}}$



- False Discovery Rate Control Strategy: applying Benjamini-Hochberg algo onto $\widehat{\alpha}_i$s.

Ratio $S^{\text{LASSO}} / S^{\text{OPT}}$

$\rho \times 100$ \ $\mu/\sigma \times \sqrt{12}$	10.95	8.22	5.48	4.38	3.29	2.74	2.19	1.92	1.64	1.37	1.10	0.99	0.88	0.77	0.66	0.55	0.44	0.33	0.22	0.16	0.11
50.00	1.00	0.99	0.98	0.98	0.97	0.97	0.98	0.98	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
25.00	1.00	0.99	0.98	0.97	0.97	0.97	0.98	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
20.00	1.00	0.99	0.98	0.97	0.97	0.97	0.98	0.99	0.99	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
15.00	1.00	0.99	0.98	0.97	0.96	0.97	0.98	0.99	0.99	0.99	0.99	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
10.00	1.00	0.99	0.98	0.97	0.96	0.97	0.98	0.99	0.99	0.98	0.98	0.98	0.98	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00
7.50	1.00	0.99	0.98	0.97	0.96	0.97	0.98	0.99	0.99	0.98	0.97	0.97	0.98	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00
5.00	1.00	0.99	0.98	0.97	0.96	0.97	0.98	0.99	0.98	0.97	0.96	0.96	0.97	0.98	0.98	0.99	1.00	1.00	1.00	1.00	1.00
2.50	1.00	0.99	0.98	0.96	0.96	0.97	0.98	0.98	0.97	0.95	0.94	0.95	0.96	0.97	0.98	0.99	1.00	1.00	1.00	1.00	1.00
1.25	1.00	0.99	0.97	0.96	0.96	0.97	0.98	0.98	0.96	0.93	0.92	0.93	0.95	0.96	0.98	0.99	0.99	1.00	1.00	1.00	1.00
1.00	1.00	0.99	0.97	0.96	0.96	0.97	0.98	0.97	0.96	0.93	0.92	0.93	0.94	0.96	0.98	0.99	0.99	1.00	1.00	1.00	1.00
0.75	1.00	0.99	0.97	0.96	0.96	0.97	0.98	0.97	0.95	0.92	0.91	0.92	0.94	0.96	0.98	0.99	0.99	1.00	1.00	1.00	1.00
0.50	1.00	0.99	0.97	0.96	0.96	0.97	0.98	0.97	0.94	0.90	0.90	0.92	0.94	0.96	0.97	0.99	0.99	1.00	1.00	1.00	1.00
0.25	1.00	0.99	0.97	0.96	0.96	0.98	0.97	0.95	0.92	0.88	0.89	0.91	0.93	0.96	0.97	0.99	0.99	1.00	1.00	1.00	1.00
0.12	1.00	0.99	0.97	0.95	0.96	0.98	0.97	0.94	0.89	0.86	0.89	0.91	0.93	0.95	0.97	0.99	0.99	1.00	1.00	1.00	1.00

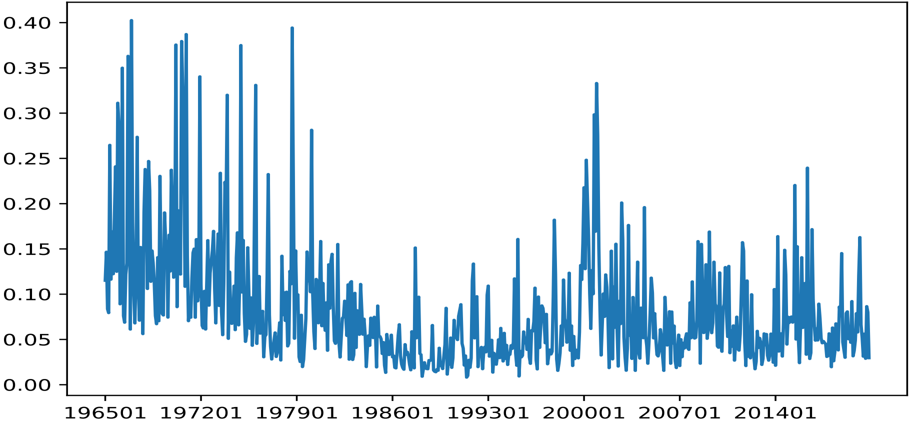
Colour scale: 0.88, 0.9, 0.92, 0.94, 0.96, 0.98

- Lasso Strategy: applying Lasso algo onto $\widehat{\alpha}_i$s.

Empirical Analysis of US Equities

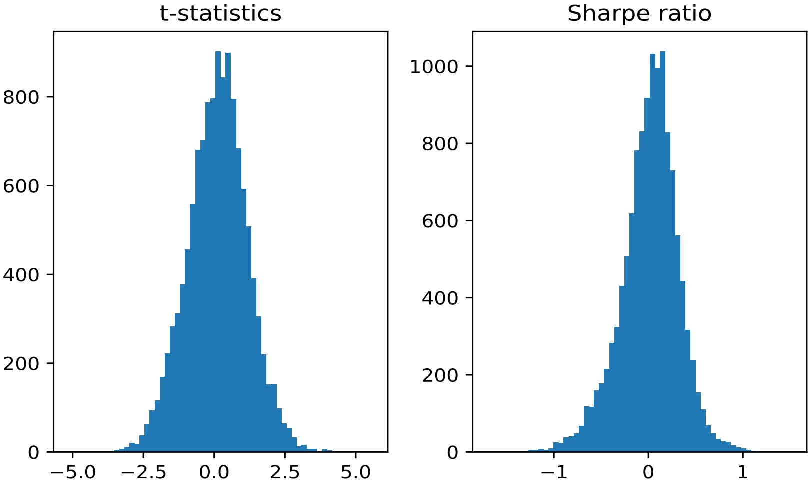
- We study US monthly equity returns from January 1965 to December 2020.
- We adopt a multi-factor model with 16 characteristics and 11 GICS sectors, including market beta, size, operating profits/book equity, book equity/market equity, asset growth, momentum, short-term reversal, industry momentum, illiquidity, leverage, return seasonality, sales growth, accruals, dividend yield, tangibility, and idiosyncratic risk.
- 10-year rolling window estimation, last 2 years as validation sample for tuning parameter selection

Time-series of the Cross-sectional R^2s



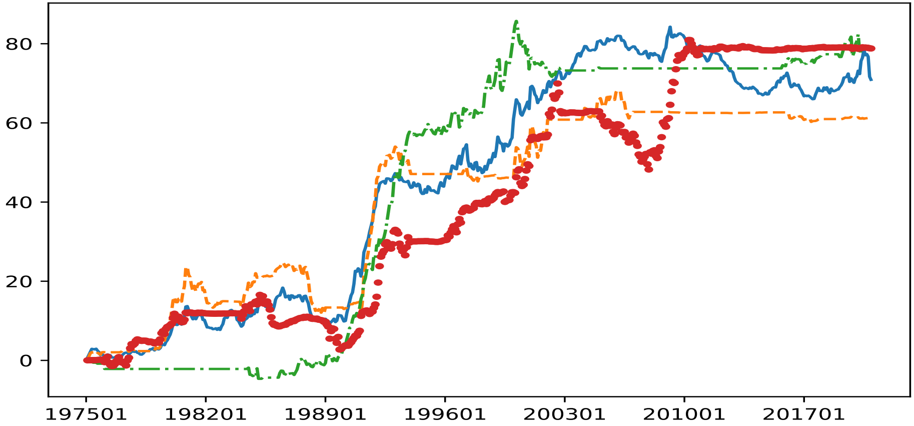
Lewellen (2015, CFR): 7.8%, 1964/05 - 2009/12; Gu, Kelly, and Xiu (2021, JoE): 12-14% 1987 - 2016

Rare and Weak Alphas



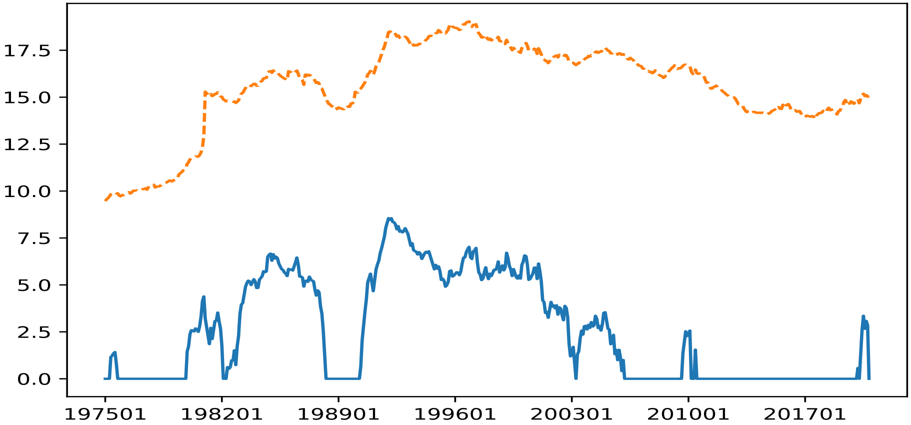
6.35% (0.63%) of the t-Stats > 2.0 (3.0); 0.505% alphas with a Sharpe ratio > 1.0.

Performance of Risk-Normalized Arbitrage Portfolios



Sharpe Ratios: OPT (red, 0.496), CSR (blue, 0.450), BH (green, 0.497), and LASSO (orange, 0.384)
In contrast, average of $\widehat{S}^{\star}$ is about 2.95, which is far greater than feasible Sharpe ratios, ~ 0.5.

Biased vs. Consistent Sharpe Ratio Estimators



$\widehat{S}^\star$ (blue) and $\widetilde{S}^\star$ (orange)

Average of $\widehat{S}^\star$ is about 2.95, which is far greater than feasible Sharpe ratios, ~ 0.5.

Conclusion

- Statistical limit to arbitrage: Widens the bounds in which mispricing can survive in presence of arbitrageurs.
- Existing empirical evidence provides "lower bound" of Sharpe ratios achievable with machine learning methods (based on ad-hoc choices). Our theoretical analysis provides an "upper bound" in a specific context (based on optimal strategy).
- The gap between feasible and infeasible Sharpe ratios will further increase if arbitrageurs face additional statistical challenges, e.g., model misspecification, omitted factors, weak factors, large non-sparse covariance matrix.