Parallax and Tax*

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October 11, 2019

Abstract

Common valuations pose an obstacle to trade and, hence, an existential threat to brokers, who profit from taxing trade. We write down a model in which a broker drives a wedge between valuations by deceiving gullible traders. Separate valuations facilitate arbitrage, and arbitrage generates brokerage fees. We then show that this process, which we call *parallax and tax*, explains a classic case of market inefficiency: the favorite-longshot bias in horserace parimutuel markets.

Keywords: deception, arbitrage, favorite-longshot bias, neural network, structural estimation

JEL Classification: D22, D82, D83, L83

*Please direct correspondence to: etangr@wharton.upenn.edu. We thank Bosko Blagojević and Will Cai for help scraping and parsing the data; Joe Appelbaum, Hamsa Bastani, Gerard Cachon, Bo Cowgill, Stefano DellaVigna, Botond Köszegi, Dorothy Kronick, David Laibson, Simone Marinesi, Cade Massey, Ken Moon, Dave Pennock, Alex Rees-Jones, Bernard Salanié, Michael Schwert, Ricardo Serrano-Padial, Joe Simmons, Erik Snowberg, Ran Spiegler, Richard Thaler, and Eric Zitzewitz for helpful conversations and comments on previous drafts; seminar audiences at UC Berkeley, Wharton, and SITE; and Wharton and Microsoft Research for generous financial support. A previous version of this paper was titled, "The Favorite-Longshot Midas."

Profit is sweet, even if it comes from deception.

Sophocles

1 Introduction

Inefficiencies characterize important financial markets. In the stock market, excess returns are persistent, rather than fleeting (Mehra and Prescott, 1985, 2003); prices swing wildly, even in the absence of news (LeRoy and Porter, 1981; Shiller, 1981, 1992; Shleifer, 1986); and price movements are predictable over short and long horizons (Fama and French, 1988; Jegadeesh and Titman, 1993, 2011).[1]

Market inefficiencies stem in part from the presence of irrational traders (Shiller, 1984; Shleifer and Summers, 1990; De Long et al., 1991). These "noise traders" rely on faulty heuristics and take advice credulously, and they misprice securities by trading on their systematically biased beliefs. Although arbitrage by more sophisticated investors attenuates the mispricing, various limits, such as transaction costs, illiquidity, and risk, preserve some degree of inefficiency (for a review, see Gromb and Vayanos, 2010).

A defining trait of noise traders is their gullibility (Akerlof and Shiller, 2015). In pump-and-dump schemes, for example, conspirators deceive gullible traders into overvaluing an asset (i.e., pump) so as to sell their own holdings at higher prices (i.e., dump). We characterize a "parallax-and-tax" scheme in which a third-party broker, rather than an investor, deceives noise traders and then profits by taxing trade, rather than trading assets. In a visual parallax, the same object appears in different locations when viewed from different positions; here, a broker induces different valuations of the same asset from different investors. Common valuations pose an obstacle to trade (Tirole, 1982) and, hence, an existential threat to brokers. Deception drives a wedge between valuations, increasing trade and brokerage fees.

This paper models a parallax-and-tax scheme and uses the model to rationalize a classic case of market inefficiency. The model applies to a brokered market with informed arbitrageurs and credulous noise traders. We begin with an illustrative version in which the broker offers two securities: a *favorite* that pays out if a probable event occurs, and a *long-shot* that pays out if the event does not occur; later we extend the model to an arbitrary number of securities. The true probability of the event, $p \geq \frac{1}{2}$, is known to the broker and the arbitrageurs but not to the noise traders.

[1] For reviews, see Barberis and Thaler (2003) and Shiller (2003).

The broker charges separate fees to arbitrageurs and noise traders. It also announces a prediction, q, which noise traders adopt as their belief. Noise traders invest in the first period. If $q < p$, they over-invest in the longshot, creating a favorite-longshot bias: expected returns for the favorite exceed those for the longshot. In the second period, arbitrageurs partially correct this bias by investing in the favorite. Some bias remains, however, because arbitrageurs pay brokerage fees.

We show that under optimal price discrimination, the residual favorite-longshot bias is increasing in the broker's deception. Larger lies induce noise traders to more severely misprice the securities, creating a larger arbitrage opportunity. This allows the broker to charge higher fees to arbitrageurs, thereby limiting their capacity to correct the bias.

In light of this result, we revisit the favorite-longshot bias in horserace parimutuel markets (Griffith, 1949; Thaler and Ziemba, 1988). In parimutuel markets, bettors place wagers on outcomes (e.g., a given horse winning the race), the broker takes a tax, and the remainder is divided among winning wagers. If bettors are risk-neutral and have correct beliefs, then expected returns should be the same for favorites and longshots. Consistent with previous literature, we find that wagers on longshots return far less in expectation than wagers on favorites. Whereas a favorite with 1/1 odds (i.e., that pays a \$1 dividend on a winning \$1 wager) returns 85 cents on the dollar on average, a longshot with 30/1 odds (i.e., that pays a \$30 dividend on a winning \$1 wager) returns just 63 cents on the dollar.

Three explanations have been offered for the favorite-longshot bias: 1) some bettors are risk-loving (e.g., Weitzman, 1965), 2) some bettors overweight small probabilities (e.g., Griffith, 1949), and 3) some bettors wager randomly (e.g., Hurley and McDonough, 1995).[2] We document a new stylized fact that is inconsistent with these explanations: the extent of the favorite-longshot bias varies across tracks.[3] At some tracks, expected returns for favorites greatly exceed those for longshots. At other tracks, returns for favorites and longshots do not differ in expectation. Explanations that rely on risk preferences, cognitive biases, or random wagering cannot rationalize this pattern without sizable, and apparently idiosyncratic, cross-track variation in risk preferences, cognitive biases, or random wagering.

Instead, we show that the extent of the favorite-longshot bias is predicted by variation in the accuracy of the track's predictions. Tracks provide bettors with a prediction for each horse, in the form of odds. These *morning-line odds* have no formal bearing on the

[2]For a comprehensive review of the explanations offered for the favorite-longshot bias, see Ottaviani and Sørensen (2008).

[3]We know of only one paper that examines track-specific returns (Swidler and Shaw, 1995), which studies a single track in Texas—and finds no favorite-longshot bias.

parimutuel odds. Instead, their ostensible purpose, according to oddsmakers, "is to predict, as accurately as possible, how the betting public will wager on each race"—i.e., to predict the final parimutuel odds.[4]

Past studies of the favorite-longshot bias in horserace parimutuel markets ignore the morning-line odds, perhaps because they are difficult to obtain or because their formal irrelevance suggests economic irrelevance.[5] We scrape the morning lines each morning for over a year, and we find that these predictions exhibit a favorite-longshot bias: on average, they are insufficiently short for favorites and insufficiently long for longshots.[6] Horses with 1/1 morning-line odds begin the race with parimutuel odds of 1/2, on average, while horses with 30/1 morning-line odds end up with parimutuel odds longer than 50/1. The morning-line odds provide useful ordinal information by correctly distinguishing between longshots and favorites. However, they provide biased cardinal information by underestimating the odds of longshots and overestimating the odds of favorites.

Across tracks, the extent of the miscalibration in the morning-line odds predicts the extent of the favorite-longshot bias in the parimutuel odds. Some tracks provide well-calibrated morning-line odds, and parimutuel odds at those tracks do not exhibit a favorite-longshot bias. Other tracks embed a severe favorite-longshot bias in the morning-line odds, and parimutuel odds at those tracks reflect a severe favorite-longshot bias.

These stylized facts, along with two features of the institutional context, motivate an application of our two-period brokerage model with noise traders and arbitrageurs. First, online wagering allows sophisticated bettors to wager near the close of the betting window, when live parimutuel odds best predict parimutuel odds at closing. As we show, late wagers concentrate on favorites. And as we discuss, journalistic accounts attribute these late wagers to professional gamblers. Second, tracks offer steep discounts to volume bettors.

We adapt our two-period model to predict final parimutuel odds and expected returns from the morning-line odds. In the first period, noise traders infer beliefs, q, under the assumption that the morning-line odds reflect wagering by risk-neutral bettors. Noise traders then wager in proportion to these beliefs, generating odds that reproduce any bias inherent

[4] Quoted in *Getting a Line on the Early Odds*, by Joe Kristufek. `https://web.archive.org/web/20161028154509/www.arlingtonpark.com/node/10222`. Similarly, the horse racing columnist Bob Ehalt writes, "The intent [of the morning lines] is to frame how the betting on the race will unfold." `https://www.americasbestracing.net/gambling/2018-morning-line-odds-betting-tool`. Also see: Snyder (1978, 1117).

[5] For instance, Ottaviani and Sørensen (2009, 2129) justify studying racetrack parimutuel markets because of "the absence of bookmakers (who could induce biases)."

[6] To our knowledge, the only other researcher to collect the morning lines is Snyder (1978), who collects them by hand and puzzles over the fact that they reflect a favorite-longshot bias.

in the morning lines. In the second period, arbitrageurs infer beliefs, $p(q)$, from the observed rates at which horses with implied probability q actually win. Given that the morning-line odds embed a favorite-longshot bias, this correction leads arbitrageurs to view favorites as underpriced—and after the discount, often dramatically so. The track sets the discount to maximize its own income, and a large number of arbitrageurs place all wagers with positive expected value until no more exist. Late wagering concentrates on favorites, and the favorite-longshot bias moderates. The bias does not disappear, however, because arbitrageurs pay some tax.

Previous empirical studies of the favorite-longshot bias in horserace betting markets attempt to adjudicate between preference and belief-based explanations. In this literature, a model is proposed which allows for a downward-sloping relationship between the parimutuel odds and expected returns. At least one parameter is tuned to match the data. Parameter estimates are then interpreted as evidence of risk-loving preferences (Weitzman, 1965; Ali, 1977; Golec and Tamarkin, 1998), a bias towards overweighting small probabilities (Snowberg and Wolfers, 2010), or heterogeneity in beliefs (Gandhi and Serrano-Padial, 2014) or preferences (Chiappori, Salanié, Salanié and Gandhi, 2019). Chiappori et al. (2019) conclude that probability weighting is needed to match the data.

By comparison, our model is restricted in two ways. First, it predicts returns from the formally irrelevant morning-line odds, not the final parimutuel odds. Second, our model has zero degrees of freedom. All agents are risk-neutral, and none transforms probabilities. Despite these handicaps, our model closely predicts the observed relationship between odds and returns—and it predicts the variation in that relationship across tracks.

In our model, biased morning lines generate incremental income for the track. Whereas state-regulated tax rates cap the losses that noise traders should sustain in expectation, misinformed traders overbet losers and sustain excess losses. Because arbitrage is assumed to be competitive, these excess losses flow in their entirety into the track's coffers, laundered through taxes on arbitrageurs. Risk-neutral arbitrageurs absorb the risk, while the broker extracts the profits. For tracks that embed a favorite-longshot bias in the morning-line odds, we estimate that misleading noise traders increases brokerage income by as much as 25%.[7]

The remainder of the paper is organized as follows. Section 2 discusses related research. Section 3 presents our model of a brokered market with two parimutuel securities. Section 4 describes the empirical context and the data. Section 5 illustrates the favorite-longshot

[7]Other taxes on arbitrageurs, such as fees for data (e.g., Budish, Lee and Shim, 2019), are outside of our model.

bias and other stylized facts. Section 6 generalizes our model for an arbitrary number of securities, Section 7 compares theoretical predictions with the data and estimates each track's incremental income from misinformation. In Section 8, we consider the broker's optimal deception strategy in the context of our two-security model. Section 9 concludes.

2 Related literature

Our model extends a class of models showing that heterogeneous beliefs can generate a favorite-longshot bias in parimutuel markets (Borel, 1938; Ali, 1977; Hurley and McDonough, 1995; Ottaviani and Sørensen, 2009, 2010; Gandhi and Serrano-Padial, 2014).[8] The simplest versions of these models allow for two types of bettors: arbitrageurs, who know the true probabilities, and noise traders, whose beliefs follow an arbitrary distribution. Our approach endogenizes the beliefs of noise traders, rather than treating them as a primitive (cf. Kyle, 1985; Black, 1986). In our model, a broker induces heterogeneous beliefs by misleading noise traders.

A number of theoretical papers analyze deception by firms (e.g., Gabaix and Laibson, 2006; Heidhues, Kőszegi and Murooka, 2016). For instance, Gabaix and Laibson (2006) model a firm that shrouds add-on costs, such as high-priced printer toner. In these models, profits on naive customers (who buy the toner) offset losses on sophisticated customers (who only buy the printers). Here, by contrast, the broker profits from both naifs and sophisticates.

A similar distinction applies to fixed-odds betting markets, in which a bookmaker sets the odds. In fixed-odds markets, bookmakers generate excess returns by taking risky positions against bettors with inaccurate beliefs, while using the tax to limit betting by "sharps" (Levitt, 2004). In our model, the track encourages the sharps to participate and generates a riskless profit by taxing them.[9]

Models of deception offer a new direction for structural work in behavioral economics, which has focused on models of behavioral biases, such as hyperbolic discounting and reference dependence (for a review, see DellaVigna, 2018). We provide a template for interactions between gullible and knowing agents that can be taken directly to data. Naive agents are assumed to believe what they are told, and sophisticated agents exploit their gullibility.

Favorite-longshot biases appear in other markets. In prediction markets where transaction costs restrict shorting longshots, longshots tend to be overpriced (Wolfers and Zitzewitz,

[8]This result can also be generated in markets for Arrow-Debreu securities, which unlike parimutuel securities have a variable price and a fixed dividend (Manski, 2006; Ottaviani and Sørensen, 2015).

[9]For models of pricing in fixed-odds markets, see: Shin (1991, 1992); Ottaviani and Sørensen (2005).

2004, 2006); in prediction markets without such costs, longshots (and favorites) are priced efficiently, or nearly so (Servan-Schreiber et al., 2004; Cowgill et al., 2009; Cowgill and Zitzewitz, 2015; Dana et al., 2019). In markets for stock index options, out-of-the-money (i.e., longshot) put options are overpriced, consistent with hedging, as are strongly out-of-the-money call options, consistent with speculation (Tompkins, Ziemba and Hodges, 2008). In fixed-odds betting markets—in which a bookmaker sets the odds—returns for favorites outpace returns for longshots in British horserace betting (Williams and Paton, 1997; Jullien and Salanié, 2000; Snowberg and Wolfers, 2010), but the reverse is generally true for American team sports (Gandar et al., 1988; Woodland and Woodland, 1994; Levitt, 2004; Simmons et al., 2010). Here, biases in the odds are decided by the bookmaker, not the market. We conclude that favorite-longshot biases appear in different markets for different reasons.

To our knowledge, morning lines odds do not exist in horserace parimutuel markets outside of North America. In Hong Kong, Japan, and New Zealand, tracks provide predictions in the form of ratings, rather than odds.[10] In each of these countries, the final parimutuel odds do not exhibit a favorite-longshot bias (Busche and Hall, 1988; Busche, 1994; Gandar et al., 2001).[11] In Britain and Australia, parimutuels compete with fixed odds, which may influence naive bettors in the same manner as the morning-line odds. Given that the fixed odds exhibit a favorite-longshot bias in Britain (see above), we suspect that parimutuel odds do as well, though we know of no empirical study of parimutuel odds in these countries.

3 Theoretical framework

A broker offers two parimutuel securities whose returns are pegged to a binary event. One security pays off if the event occurs; the other pays off if the event does not occur. Let the probability of the event be $p \geq \frac{1}{2}$. Since the event is (weakly) more likely to occur than not, the security tied to the event is the favorite, and the other security is the longshot. The broker announces a prediction for the favorite, q, which some investors take at face value.[12] In the first period, these noise traders act on their mistaken beliefs and misprice the

[10] For Hong Kong race cards, see: `https://racing.hkjc.com/racing/Info/meeting/RaceCard/english/Local/`. For Japan race cards, see: `http://japanracing.jp/en/racing/result/`. For New Zealand race cards, see: `https://loveracing.nz/RaceInfo.aspx#bm-meeting-calendar`.

[11] We know of no empirical study of horserace betting markets in France, which are also run as parimutuels.

[12] The broker's prediction is cheap talk. We do not write down a cheap talk signaling game (e.g., Crawford and Sobel, 1982) because cheap talk games cannot rationalize biased signals, which we observe in our data. In a cheap talk signaling game, the receiver knows the bias and subtracts it out. Instead, we assume that some investors are gullible. This is similar in spirit to a cursed equilibrium (Eyster and Rabin, 2005), in which agents ignore the signal, though diametrically opposed in practice—here, agents ignore their prior.

securities. Arbitrageurs, who know p, partially correct the inefficiency in the second period.

The broker charges separate fees to noise traders and arbitrageurs, and we show that under optimal price discrimination, greater deception by the broker generates larger inefficiencies in the market. (We consider the broker's choice of q in Section 8.)

3.1 Set-up

In a parimutuel market, investors purchase shares of a security for a set price (e.g., \$1), the broker takes a fee, σ, and the remaining investment is split among shareholders of the security that pays out. Let s be the share of investment in the favorite. If the event occurs, each dollar invested in the favorite returns $(1-\sigma)/s$, and investments in the longshot return 0. If the event does not occur, each dollar invested in the longshot returns $(1-\sigma)/(1-s)$, and investments in the favorite return 0. The more invested in a security, the lower its returns.

We write down a two-period model with two types of risk-neutral investors. Noise traders invest in the first period; arbitrageurs invest in the second. Noise traders are gullible. Absent beliefs about p, they take the broker's prediction, q, at face value—i.e., they believe that the event will occur with probability q. Noise traders are also myopic: they fail to anticipate how the parimutuel returns will change in the second period.[13]

In the first period, noise traders invest in the security with the higher expected value according to their adopted beliefs, q. Given the brokerage fee, σ, both securities may portend losses. We rationalize the decision to invest with an additive utility shock, u, consistent with a direct utility from gambling (Conlisk, 1993), and we allow the shock to vary across noise traders. Thus, noise trader j values a \$1 investment in the favorite at $q(1-\sigma)/s + u_j$ and a \$1 investment in the longshot at $(1-q)(1-\sigma)/(1-s) + u_j$. A large number of noise traders each decide whether to invest \$1, and if so, which security to purchase. We are interested in s^*, the Walrasian equilibrium share of first-period investment in the favorite.

In the second period, arbitrageurs invest in the security with excess returns. Let x be the amount invested by arbitrageurs in the favorite, and let y be the amount invested by arbitrageurs in the longshot. Normalizing the total investment by noise traders to 1, total investment after arbitrage is $1 + x + y$, and final parimutuel returns on a \$1 investment are

[13] It is straightforward to write down an alternative model in which noise traders correctly forecast second-period investment. In such a model, noise traders invest entirely in the longshot if $q < p$ and entirely in the favorite if $q > p$; arbitrageurs invest in the other security; arbitrage volume is higher; and broker income is higher as a result. We view this model as implausible because it requires noise traders to know how q compares to the truth—and to invest according to q anyway. In other words, one cannot be gullible without also being myopic.

$(1-\sigma)(1+x+y)/(s^*+x)$ for the favorite and $(1-\sigma)(1+x+y)/(1-s^*+y)$ for the longshot.

Unlike noise traders, arbitrageurs know p, and they do not receive direct utility from investing. Their expected return on a \$1 investment in the favorite is:

$$\mathbb{E}\big[V(\$1 \text{ in favorite})\big] = p(1-\sigma)\frac{1+x+y}{s^*+x}$$

And their expected return on a \$1 investment in the longshot is:

$$\mathbb{E}\big[V(\$1 \text{ in longshot})\big] = (1-p)(1-\sigma)\frac{1+x+y}{1-s^*+y}$$

The broker price discriminates by offering arbitrageurs a rebate r on each dollar invested, regardless of the outcome. We assume that σ is fixed, as when brokers perfectly compete for noise traders or when its fees are regulated. However, the broker holds a monopoly over its own securities, and as such, selects r.

We assume that arbitrage is competitive. Hence, the second-period equilibrium is defined by arbitrage volumes $\{x, y\} \geq 0$ and a rebate r such that:

1. Arbitrageurs make zero profits in expectation.
   - $\mathbb{E}\big[V(\$1 \text{ in favorite})\big] = \$1 - r$ when $x > 0$
   - $\mathbb{E}\big[V(\$1 \text{ in longshot})\big] = \$1 - r$ when $y > 0$
2. Arbitrageurs leave no money on the table.
   - $\mathbb{E}\big[V(\$1 \text{ in favorite})\big] \leq \$1 - r$ when $x = 0$
   - $\mathbb{E}\big[V(\$1 \text{ in longshot})\big] \leq \$1 - r$ when $y = 0$

3.2 Equilibria

In the first-period Walrasian equilibrium, $s^* = q$, and only those for whom $u_j > \sigma$ invest. Consider an example with $\sigma = 0.2$ and $p = 0.6$. If the broker predicts $q = 0.5$, noise traders split their investment evenly among the securities. For a \$1 investment on either the favorite or the longshot, the parimutuel advertises a return of $(1-\sigma)/q = (1-\sigma)/(1-q) = \$1.60$ were the security to pay out, and noise traders expect a return of $1 - \sigma = 80$ cents. The true expected returns, however, are $p(1-\sigma)/q = 96$ cents for the favorite and $(1-p)(1-\sigma)/(1-q) = 64$ cents for the longshot. Noise traders are gullible, and deception by the broker induces them to misprice the securities.

Let γ_1 be the ratio of first-period expected returns for the favorite and longshot under correct beliefs:

$$\gamma_1 = \frac{p(1-q)}{q(1-p)}$$

If $q < p$, the expected return on the favorite exceeds the expected return on the longshot—i.e., a favorite-longshot bias. And if $q > p$, the expected return on the longshot exceeds the expected return on the favorite—i.e., a reverse favorite-longshot bias.

In the second period, arbitrageurs invest in the security that promises excess returns. If the rebate is sufficiently large,[14] arbitrageurs invest in the favorite under a favorite-longshot bias, and they invest in the longshot under a reverse favorite-longshot bias. In both cases, arbitrageurs drive down the security's expected returns to $1-r$ for every dollar invested, regardless of q. Hence, for a *fixed* (and sufficiently large) rebate r, the extent of the second-period bias depends only on the size of the rebate.

Now consider the rebate r^* that maximizes the broker's income:

$$r^* = \arg\max_r \sigma + (\sigma - r)(x+y)$$

For each dollar invested by noise traders, the broker takes a share σ and arbitrageurs invest $x+y$ dollars, of which the broker takes a share, $\sigma - r$.

The optimal brokerage fee on arbitrage is:

$$\sigma - r^* = (1-\sigma)\begin{cases} (1-p)\big(\sqrt{\gamma_1}-1\big), & q \leq p \\ p\left(\frac{1}{\sqrt{\gamma_1}}-1\right), & q > p \end{cases}$$

When the broker tells the truth, $\gamma_1 = 1$, and arbitrage is free. Otherwise, fees on arbitrage are strictly positive and increasing in the first-period bias—i.e., increasing in γ_1 when $q < p$ and decreasing in γ_1 when $q > p$. Larger lies (i.e., γ_1 farther from 1) allow the broker to tax arbitrage at higher rates.

Arbitrage attenuates the first-period bias. Let γ_2 be the ratio of second-period expected returns for the favorite and longshot under correct beliefs:

$$\gamma_2 = \sqrt{\frac{p(1-q)}{q(1-p)}} = \sqrt{\gamma_1}$$

[14] If $r > 1 - p(1-\sigma)/q$ when $q < p$, or if $r > 1-(1-p)(1-\sigma)/(1-q)$ when $q > p$.

Arbitrage narrows, but does not eliminate, the gap in expected returns. When $q < p$, $1 < \sqrt{\gamma_1} < \gamma_1$; when $q > p$, $\gamma_1 < \sqrt{\gamma_1} < 1$. In other words, arbitrage brings the ratio of expected returns closer to 1. Arbitrage does not equalize expected returns because arbitrageurs pay positive brokerage fees.

Under the optimal rebate, the bias becomes more extreme with the size of the broker's lie. When $q < p$, lower values of q amplify the favorite-longshot bias; when $q > p$, higher values of q amplify the reverse favorite-longshot bias. This occurs because the broker who tells a larger lie taxes arbitrage at a higher rate—and thus more greatly limits the scope for arbitrage to moderate the bias.

4 Context and data

We apply our model to betting markets at North American horse races, which are run as parimutuels. As in our model, bettors place wagers on outcomes (e.g., a certain horse winning the race), the track takes a state-regulated share σ known as the takeout, and the remainder is split among wagers on the winning outcome. Let s_i denote the share of wagers placed on outcome i. A \$1 wager on i returns $(1 - \sigma)/s_i$ if i occurs, and it returns 0 if i does not occur. Prospective returns are represented as odds, or as the dividend paid on a winning \$1 wager: $(1-\sigma)/s_i - 1$.[15] For example, a winning bet at 2/1 odds pays a \$2 dividend for every \$1 wagered, along with the principal. Tracks update the parimutuel odds in real time on a central monitor called the tote board, as well as online, but only the final odds, or those the start of the race, are used to calculate payoffs.

Wagers on different types of outcomes are collected in separate pools, and odds are calculated within each pool. All tracks offer win, place, and show pools, in which bets are placed on individual horses, and wagers pay out if the horse finishes first (win), first or second (place), or in the top 3 (show). Tracks also offer a selection of "exotic" pools, in which payoffs are contingent on multiple outcomes. Exacta, trifecta, superfecta, and hi-5 pools pay out if the bettor correctly predicts the first two, three, four, or five horses, respectively, in order; quinella pools pay out if the bettor correctly predicts the first two horses in any order; and daily-double, pick-3, pick-4, pick-5, and pick-6 pools pay out if the bettor correctly predicts the winners of two, three, four, five, or six races in a row, respectively.[16]

[15] In practice, tracks round odds down (to multiples of 5 cents for small odds and to larger multiples for higher odds), a practice known as breakage that slightly increases the track's effective takeout.

[16] Many exotic wagers can also be constructed as a parlay, or contingent series, of wagers in other pools. For instance, a daily-double wager is logically equivalent to placing the full wager on a win bet in the first

Popular accounts of horserace bettors support a simple typology of sophisticates and naifs. Some bettors receive a volume-based rebate. Others pay the full tax. Some bettors place their wagers electronically. Others line up at the track. Some bettors wait until the minutes or seconds before the race begins to place their wagers—i.e., when the live odds best predict the final odds (Ottaviani and Sørensen, 2006; Gramm and McKinney, 2009). Others place their wagers twenty minutes before the race begins. Some bettors train algorithms on expensive databases of race histories. Others do not have well-formed beliefs. Anecdotally, those who receive large rebates, who wager electronically, who time their bets strategically, and who analyze data tend to be the same bettors.[17]

Tracks contract with Advance Deposit Wagering companies to recruit volume bettors and administer rebates. Volume bettors generate incremental income for the tracks, which split this income with the ADWs. Some ADWs are owned by a track or a parent company that owns multiple tracks (e.g., TwinSpires, owned by Churchill Downs Incorporated). Others are independently owned and contract with multiple tracks. In 2012, 17 ADWs were in existence, though a contemporaneous report suggested that dozens more were formed around that time. To our knowledge, the rebates are unregulated, save for the constraint that they cannot be negative—i.e., volume bettors cannot be charged more than the state-regulated takeout rates.[18]

Most empirical studies of the favorite-longshot bias in horse-racing markets analyze data from the race chart, which summarizes the outcome of a race. We compile a similar dataset by collecting: 1) the final odds for each horse in the win pool, 2) the winning outcomes in each pool, 3) the returns to wagers on those outcomes, and 4) the total amount wagered in

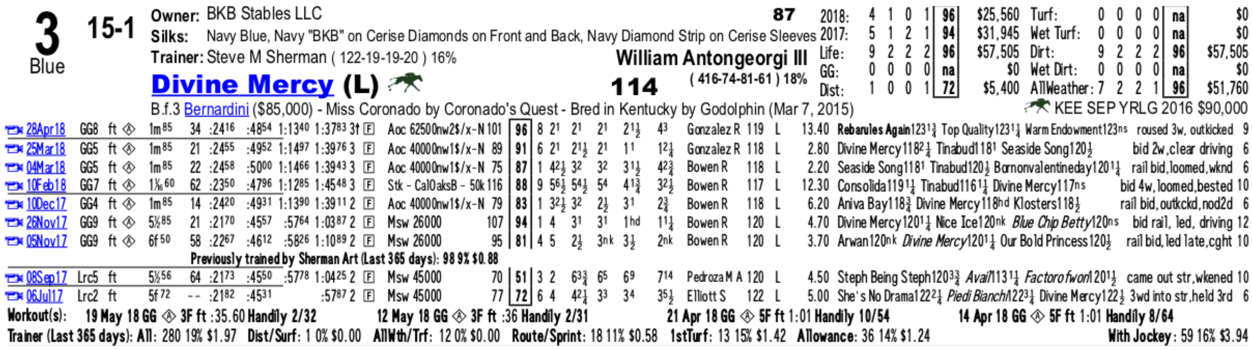
race and then placing any winnings on a win bet in the second race. Parlays are inefficient. Though the takeout is generally higher in exotic pools than in win, place, and show pools, the takeout in any exotic pool is always less than the takeout for the equivalent parlay. For example, paying the daily-double takeout once is less expensive than paying the win takeout twice.

[17]From twinspires.com, the online betting platform owned by Churchill Downs, "Late odds changes continue to confuse and confound horseplayers...There are big players in the pools and they are called 'whales.' Some whales, not all, use computer software to handicap and make their bets. Because of the volume of the whales' play, they are given rebates by advance deposit wagering companies to stimulate more betting." `https://www.twinspires.com/blog/2018/2/26/powell-explanation-behind-late-odds-changes`. A *Bloomberg* profile on Bill Benter, whose algorithm made nearly $1 billion betting in horserace parimutuel markets, writes, "The odds change in the seconds before a race as all the computer players place their bets at the same time." `https://www.bloomberg.com/news/features/2018-05-03/the-gambler-who-cracked-the-horse-racing-code`.

[18]See the 2012 "Report of the Impact of Advance Deposit Wagering on Horse Racing and Pari-mutuel Handle in New York State (`https://www.gaming.ny.gov/pdf/09191215442609.14.12.ADWReport.pdf`) and reporting on the 2012 National Horsemen's Benevolent and Protective Association (`https://www.bloodhorse.com/horse-racing/articles/132355`).

each pool. Separately, we collect track- and pool-specific takeout rates from the Horseplayers Association of North America.[19] (We do not observe the rebates to volume bettors.)

Figure 1: An example race card entry, prominently displaying the morning-line odds (15-1).



Source: Equibase.com

We supplement these data with the morning-line odds, or the formally irrelevant win odds chosen by the oddsmaker at the track.[20] Tracks publish morning lines in three places: 1) online; 2) on the tote board at the beginning of the betting window, when betting volume is low and parimutuel odds are unstable; and 3) in the race card, a printed pamphlet handed out at the track that provides information on each horse in each of the day's races. Figure 1 shows an example race card entry. Whereas historical race charts are freely available in an online repository, historical race *cards* are not. We scraped the morning-line odds each morning for 17 months from 2016 to 2018.[21] In order to analyze the data within track, we restrict our sample to the 30 tracks for which we observe at least 5,000 horse starts. Our final dataset comprises 238,297 starts in 30,631 races.[22]

Table 1 presents summary statistics by track. On average, each track hosts 7 to 10 races per day with 6 to 9 horses per race. Roughly two-thirds of wagers (in total dollars) are placed on exotic bets. Figure A3 in Appendix A shows takeout rates by track and pool. Gambling at the track is expensive. Takeouts range from 15% to 21% in win, place, and show pools, and from 12% to 31% in exotic pools. For comparison, bookmakers typically take 9% of winnings for spread and over-under bets on team sports.

All standard errors reported in the paper—either directly in the text or in tables, or as

[19] `http://www.horseplayersassociation.org/`

[20] Morning-line odds only exist in the win pool.

[21] The scraper went offline from December 2, 2016 to July 23, 2017.

[22] This reflects the discarding of 91,727 starts in 12,164 races at 68 excluded tracks. Scraping or parsing issues led to the exclusion of an additional 4,147 races at both included and excluded tracks.

Table 1: Summary statistics by track.

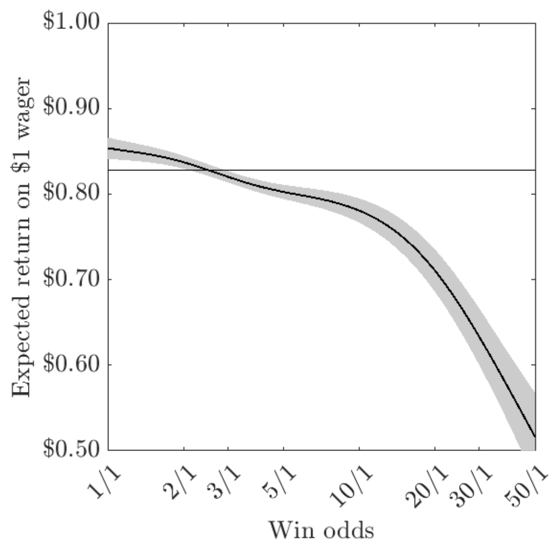
Code	Name	State	Days	Races per day	Starts per race	Exotic share
ALB	Albuquerque	NM	97	9.6	8.0	.59
AP	Arlington	IL	94	8.1	7.3	.59
AQU	Aqueduct	NY	107	8.5	7.5	.67
BEL	Belmont	NY	90	9.0	7.8	.68
BTP	Belterra Park	OH	116	8.1	7.5	.65
CBY	Canterbury	MN	92	9.4	7.9	.58
CD	Churchill Downs	KY	83	9.6	8.5	.66
CT	Charles Town	WV	197	8.2	7.7	.64
DED	Delta Downs	LA	125	10.1	8.8	.68
DEL	Delaware	DE	121	8.8	7.1	.65
DMR	Del Mar	CA	97	8.4	8.4	.65
EVD	Evangeline	LA	136	9.0	7.8	.67
FG	Fair Grounds	LA	99	9.3	8.2	.65
FL	Finger Lakes	NY	185	8.6	6.6	.67
GG	Golden Gate	CA	144	8.2	7.0	.63
GP	Gulfstream	FL	171	10.3	8.2	.70
IND	Indiana	IN	109	9.0	8.4	.68
LA	Los Alamitos	CA	119	8.4	6.7	.76
LAD	Louisiana Downs	LA	104	7.3	7.4	.67
LRL	Laurel	MD	117	9.0	7.8	.66
MNR	Mountaineer	WV	146	8.9	7.1	.65
MVR	Mahoning Valley	OH	112	8.2	8.5	.68
PEN	Penn National	PA	129	8.1	7.4	.64
PID	Presque Isle	PA	101	7.9	7.2	.66
PRX	Parx	PA	144	9.0	8.1	.67
RP	Remington	OK	86	9.6	8.9	.67
SA	Santa Anita	CA	90	8.6	8.0	.67
SUN	Sunland	NM	70	9.2	8.7	.67
TAM	Tampa Bay Downs	FL	83	9.3	8.0	.67
TUP	Turf Paradise	AZ	125	7.9	7.8	.70

95% confidence intervals in figures—are estimated from 10,000 bootstrap samples blocked by track.[23]

5 Stylized facts

Returns are systematically miscalibrated. Figure 2 shows observed returns in the win pool as a smooth function of log final odds. Wagers at all odds lose money on average, owing to the takeout. However, wagers on horses with longer odds lose more money. A $1 bet on a 1/1 favorite loses 15 cents in expectation, a smaller loss than the average takeout of 17 cents. By contrast, a 20/1 longshot loses 29 cents, and a 50/1 longshot loses 47 cents.[24] Bettors could reduce their losses by wagering on favorites instead of longshots.

Figure 2: Expected returns for win bets. The horizontal line marks one minus the average takeout.



Note: Local linear regression with a Gaussian kernel and a bandwidth, of 0.50 log odds, that minimizes the leave-one-out mean-squared error.

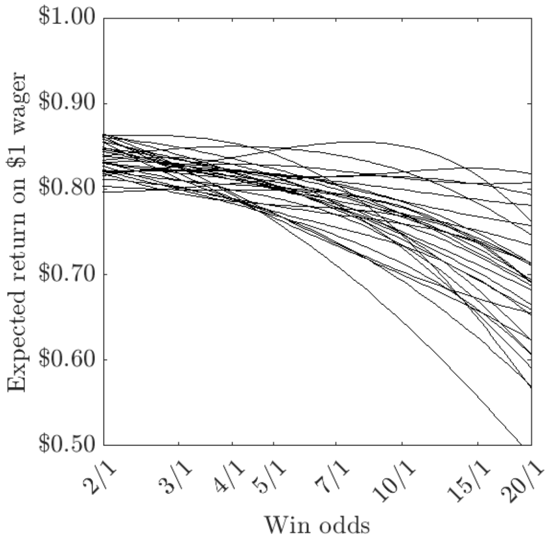
The extent of the favorite-longshot bias varies across tracks. Figure 3a shows expected returns in the win pool as a smooth function of log final odds, separately by track. (Individual

[23] We resample entire pools, rather than individual outcomes within pools. This ensures that each resampled pool retains the same number of winning outcomes as in the original data (typically 1).

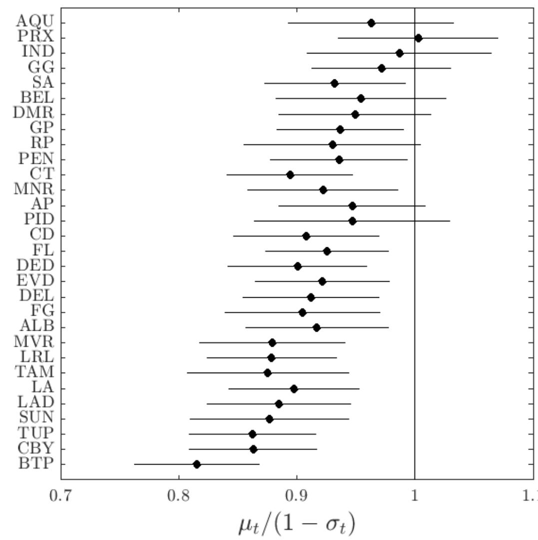
[24] These estimates are consistent with those from other analyses of parimutuel odds. For instance, Snowberg and Wolfers (2010) estimate expected returns of about 85% for a 1/1 favorite, 75% for a 20/1 longshot, and 55% for a 50/1 longshot.

Figure 3: Measures of the extent of the favorite-longshot bias across tracks.

(a) Odds and expected returns



(b) Mean returns, with 95% confidence interval



Note: In (a), we use the same bandwidth for each track, of 1.0 log odds. In (b), tracks are sorted by their expected returns at 20/1 odds in (a).

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plots are shown in Figure A2 in Appendix A.) Some tracks exhibit little or no bias, whereas others exhibit severe bias.

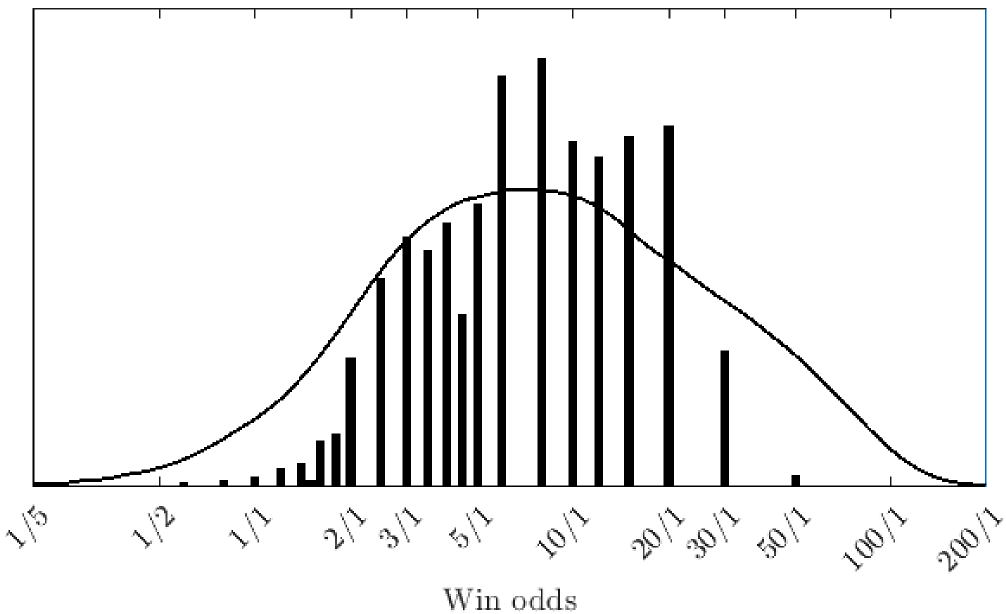
We summarize the extent of the favorite-longshot bias at track t by its mean returns, μ_t, which we calculate by averaging observed returns in the win pool. Hence, μ_t measures the expected return on a $1 wager for a bettor who picks horses uniformly at random. If horses are priced efficiently, random wagering surrenders the takeout on average, and $\mu_t = 1 - \sigma_t$. When returns on longshots are low, however, $\mu_t < 1 - \sigma_t$. This occurs because random wagering overbets longshots relative to the market.

Figure 3b shows mean returns by track, normalized by $1-\sigma_t$.[25] The tracks are ordered by their returns at odds of 20/1 in Figure 3a, which roughly orders tracks by their normalized mean returns. For some tracks, expected returns are approximately flat and normalized mean returns are close to 1, implying that wagering randomly does not generate excess losses. For the remaining tracks, estimated returns are greater for favorites than for longshots, and random wagering generates excess losses. The variation across tracks is large. A gambler

[25]In practice, the track's takeout rate is slightly higher than σ_t because of the breakage, the practice of rounding down parimutuel odds. In Figure 3b, we normalize mean returns by the true return rate, $1/\sum_{i\in\mathcal{R}} 1/(O_i+1)$.

who chooses horses randomly will lose σ_t in expectation at some tracks and as much as twice σ_t at others. This variation is also statistically significant: a Wald test rejects the null hypothesis that $\mu_t/(1-\sigma_t)$ does not vary across tracks ($p < .001$).[26]

Figure 4: Distributions of morning-line odds (bars) and final parimutuel odds (line).



Note: Distributions of log parimutuel odds estimated using a kernel density estimator with a Gaussian kernel and Silverman's rule-of-thumb bandwidth.

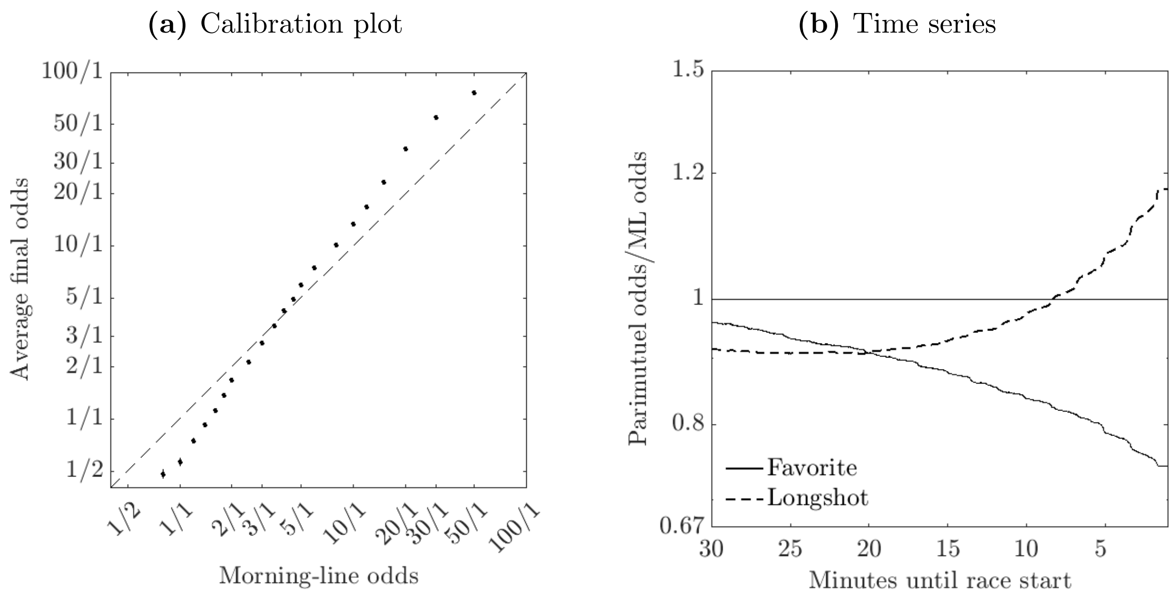
We now consider the formally irrelevant morning-line odds predicted by the track odds-maker. Figure 4 compares the distributions of morning-line and final parimutuel odds. The

[26]The returns to random wagering depend on 1) mispricing in the parimutuel odds, as shown in Figure 3a, and 2) the distribution of those odds. For instance, a track with more longshots will have lower mean returns than a track with fewer longshots, even if longshots at both tracks are overbet to the same degree. To isolate the relationship between odds and returns, we re-weight observations such that the weighted distribution of win odds at each track converges to the distribution of win odds across all tracks. For each track, we find weights, one for each observation and summing to 1, that minimizes the Kolmogorov-Smirnov test statistic—i.e., the maximal deviation—between the CDF of the weighted distribution of parimutuel odds and the CDF of the (unweighted) pooled distribution across a grid of (log) evenly spaced points. Without weighting, the KS statistic varies between 1.8pp and 7.8pp across tracks with a mean of 4.3pp. With optimal weights, it ranges from 0.1pp to 0.9pp, with a mean of 0.4pp. Let μ_t' denote the weighted mean returns at track t, which we calculate by taking a weighted average of observed returns in the win pool. Hence, μ_t' measures expected returns when sampling horses according to the pooled distribution of odds, rather than uniformly at random. As with μ_t, $\mu_t' = 1-\sigma$ when prices are efficient, and $\mu_t' < 1-\sigma$ when longshots yield lower returns than favorites. Figure A1a in Appendix A shows weighted mean returns by track, normalized by $1-\sigma_t$. Across tracks, weighted mean returns correlate with mean returns at 0.93. Again, variation across tracks is statistically significant ($p < .001$).

morning-line odds are truncated, rarely shorter than 1/1 and rarely longer than 30/1. They are also compressed: the standard deviation of the log morning-line odds is 36% smaller than the standard deviation of the log parimutuel odds. If oddsmakers are trying to predict final odds, they could do better by predicting more extreme values.

As a result, the morning-line odds fail at their ostensible goal of predicting the final parimutuel odds. Figure 5a shows average final odds at each observed morning-line odds. On average, morning lines shorter than 4/1 are insufficiently short, and morning lines longer than 4/1 are insufficiently long. For example, favorites assigned morning-line odds of 1/1 finish with final odds of 1/2 on average, and longshots assigned morning-line odds of 30/1 finish with final odds in excess of 50/1.

Figure 5: Morning-line odds and parimutuel odds.



Note: In (a), the 95% confidence intervals are obscured by the point-estimate markers for all but the shortest odds. In (b), the favorite and longshot are defined by the morning-line odds; estimates reflect second-by-second averages interpolated from scrapes at roughly 2-minute intervals, for 6,500 races.

The separation between the morning lines and the parimutuel odds is a consequence of late wagering. Figure 5b plots a time series of parimutuel odds from a separate dataset of 6,500 US races, for which we collect live odds. We plot the average log ratio of parimutuel odds to morning-line odds in the half-hour before the race, separately for the favorite and the longshot (as defined by the morning-line odds). On average, the parimutuel odds for the favorite and longshot begin near their respective morning-line odds. Thereafter, the longshot's odds lengthen and the favorite's odds shorten, and this trend accelerates as the race nears (see also: Asch, Malkiel and Quandt, 1982; Camerer, 1998). This implies that late wagers are disproportionately placed on the favorite.

The morning-line odds not only mispredict the final odds—they also imply distorted beliefs about a horse's chances of winning. If bettors are risk-neutral, then the expected value of wagering on any horse—given subjective beliefs q—must be the same for all horses in the race. That is, $q_i(O_i+1) = q_j(O_j+1)\ \forall i, j \in \mathcal{R}$, where $\mathcal{R}$ is the set of horses in a race, O denotes the final parimutuel odds, and $O+1$ is the associated return on a winning \$1 wager. Further, assume that bettors place wagers such that the parimutuel odds converge to the morning lines, denoted l, as the track oddsmaker predicts (i.e., $O_i = l_i\ \forall i \in \mathcal{R}$). Given that $\sum q_i = 1$, the implied probability of horse i winning is:

$$q_i = \frac{1}{l_i+1} \Bigg/ \sum_{j\in\mathcal{R}} \frac{1}{l_j+1} \tag{1}$$

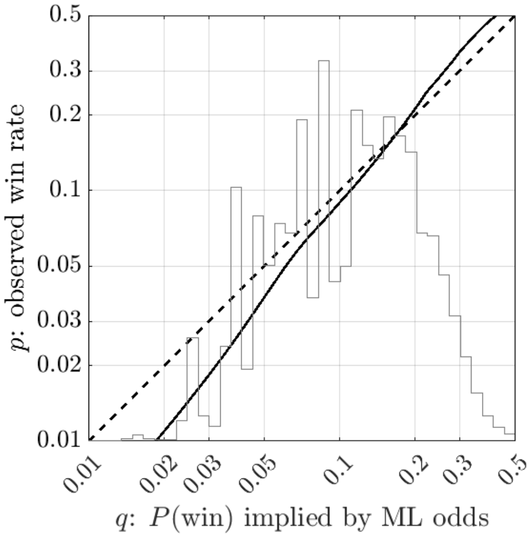
The beliefs implied by the morning-line odds (under risk-neutrality) are the inverse of the associated returns on a \$1 wager, normalized such that these beliefs sum to 1. Typically, the inverse of the normalization constant is roughly $1-\sigma_t$, implying that the morning-line odds are odds that could occur in the parimutuel.

These beliefs are misleading. Longshots underperform their implied win probabilities. Morning-line odds of 30/1 imply win probabilities of $q \approx 0.03$. Of the 7,869 starts assigned 30/1 morning-line odds, just 83, or 1.1%, won the race. Morning-line odds of 50/1 imply win probabilities of $q \approx 0.02$. Of the 588 starts assigned morning-line odds of 50/1 or longer, just 2, or 0.3%, won the race. Symmetrically, favorites outperform their implied win probabilities. Morning-line odds of 1/1 imply win probabilities of $q \approx 0.4$. Of the 557 starts assigned morning-line odds of 1/1, 56% won the race.

Figure 6 shows a calibration plot—the observed win rate (denoted by p) as a smooth function of q, the win probabilities implied by the morning-line odds. At $q \approx 0.15$, these implied beliefs are well calibrated. At other values, they are not. Implied favorites ($q > 0.15$) win more often than the morning lines predict; implied longshots ($q < 0.15$) win less often.

Let $\{p\}_{i\in\mathcal{R}}$ and $\{q\}_{i\in\mathcal{R}}$ denote the vectors of win rates and implied probabilities for a given race. We would like to estimate a function, separately by track, that predicts the vector of win rates, $\{p\}_{i\in\mathcal{R}}$, from the vector of implied probabilities, $\{q\}_{i\in\mathcal{R}}$. To do so, we use a neural network, of which multinomial logistic regression is a special case, but with the

Figure 6: Observed win rate, p, as a function of the win probability implied by the morning-line odds, q, with a histogram of q.



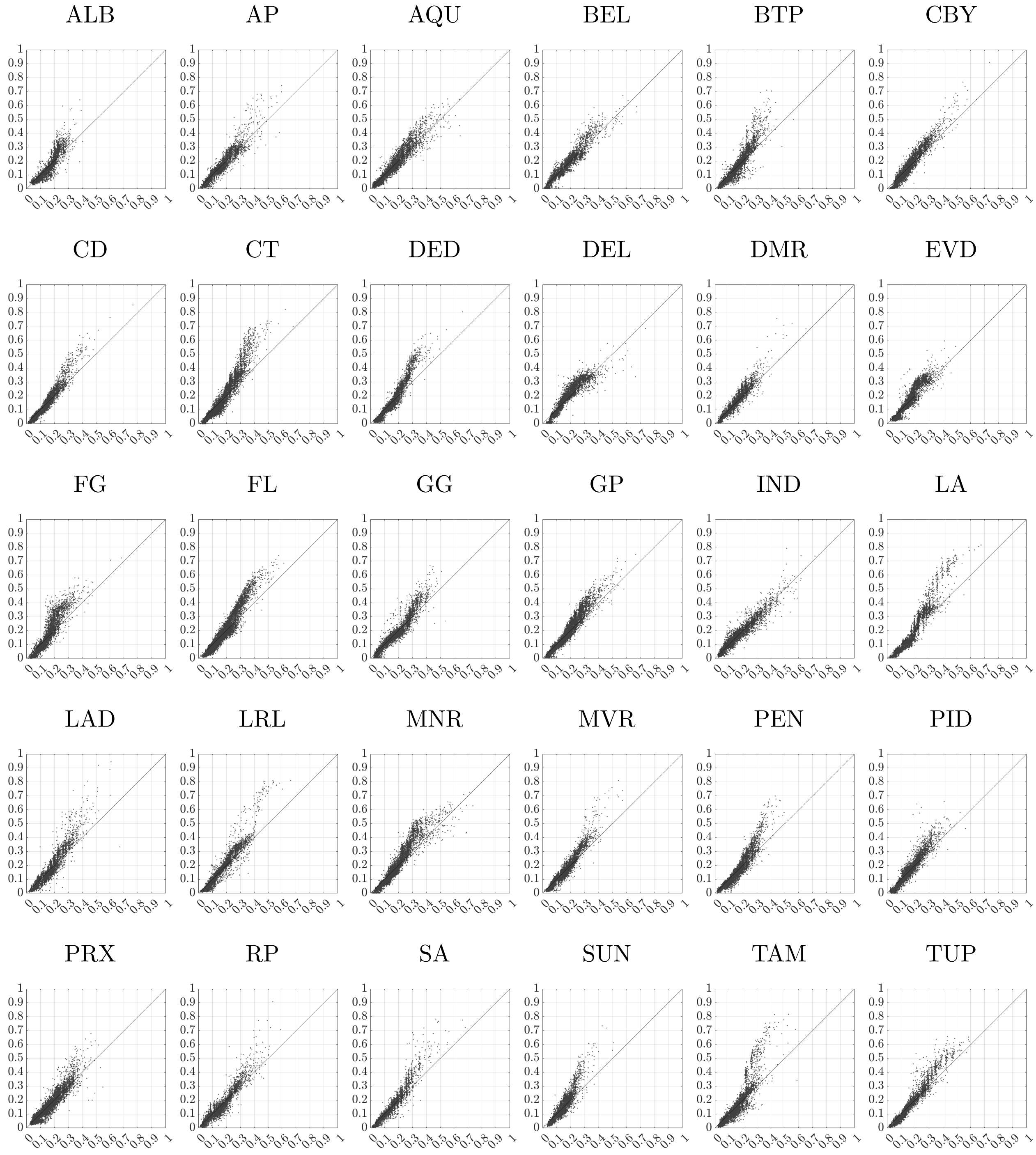
Note: Local linear regression of an indicator for winning the race on q, with a Gaussian kernel and a bandwidth of 1 percentage point.

flexibility to estimate high-dimensional, non-parametric functions. In particular, we train the neural network to predict the vector of win rates that maximizes the probabilities assigned to race winners (i.e., the likelihood). To prevent overfitting, we train using a regularizer called dropout. We also predict $\{p\}_{i\in\mathcal{R}}$ out of sample using cross-validation; for each race, we train a neural net on other races at the same track. This transformation of the implied probabilities improves predictions. The p's assign an average probability of 21.5% to the winner, compared to 19.3% for the q's, an 11% improvement. Appendix B provides more detail on the estimation routine.

The morning-line odds mislead more at some tracks than at others. Figure 7 plots p against q for each entry, separately by track. Some tracks post well-calibrated morning-line odds (e.g., IND), whereas others post miscalibrated morning-line odds (e.g., CT). Tracks that promulgate miscalibrated predictions mislead bettors in the same manner—by assigning insufficiently short morning lines to favorites (and insufficiently long morning lines to longshots). In other words, these tracks embed a favorite-longshot bias in their predictions.[27]

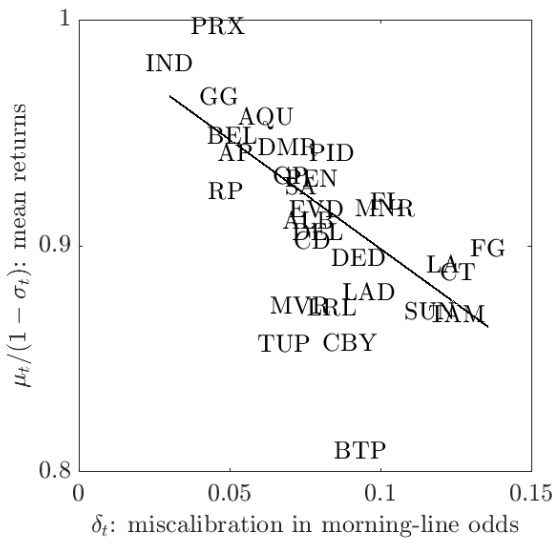
[27]The extent of the miscalibration depends on 1) how oddsmakers map true probabilities to implied probabilities (e.g., via compression or censoring) and 2) the distribution of true probabilities, which is unobserved. If, for instance, two tracks both censor longshots at 15/1, but one track races more extreme longshots, then the morning-line odds at that track will more greatly overstate the chances of longshots.

Figure 7: Predicted win rate (p) on the vertical axis as a function of the win probability implied by the morning-line odds (q) on the horizontal axis, for each entry at each track.



Note: See Appendix B for detail on the estimation procedure.

Figure 8: Scatter plot, with a regression fit line, of miscalibration in the morning-line odds and normalized mean returns, across tracks.



We summarize the extent of the miscalibration at track t using the Kolmogorov-Smirnov test statistic, which measures the maximal deviation between two empirical cumulative distribution functions. Specifically we compute the test statistic for each race as $\max_i |F(q_i) - F(p_i))|$, ordering the entries to be increasing in p, and we define δ_t as the average across races. If $q = p$ for each entry in each race, $\delta_t = 0$; otherwise $\delta_t > 0$.

Across tracks, miscalibration in the morning-line odds predicts the favorite-longshot bias. Figure 8 shows a correlation of -0.63 between δ_t and mean returns, $\mu_t/(1-\sigma_t)$, from Figure 3b.[28] The more severe the miscalibration in the morning-line odds, the more severe the favorite-longshot bias.

6 Empirical model

These stylized facts motivate an application of our model from Section 3 to horserace parimutuel markets. In this section, we generalize the model to an arbitrary number of outcomes (e.g., horses in a race). This allows us to predict an array of quantities—arbitrage volume, optimal rebates, track income, parimutuel odds, and expected returns—solely from the beliefs of noise traders and arbitrageurs, which we infer from the morning-line odds.

[28]Figure A1b in Appendix A shows a correlation of -0.61 between δ_t and weighted mean returns, $\mu'_t/(1-\sigma_t)$, from Figure A1a.

6.1 Setup

The set-up is identical to Section 3, with the exception that there are now $N \geq 2$ outcomes, indexed by i. The track announces predictions q_i such that $\sum_{i=1}^{N} q_i = 1$.

In the first period, noise traders take the track's predictions at face value and wager on the outcome with the highest subjective expected value. Noise trader j values a \$1 investment in outcome i at $q_i(1-\sigma)/(1-s_i)+u_j$, where s_i is the share of first-period wagering on outcome i and u_j is the direct utility that noise trader j receives from wagering \$1. A large number of noise traders each decide whether to wager a \$1 endowment, and if so, which outcome to bet on. We are interested in s_i^*, the Walrasian equilibrium share of first-period wagering on each outcome.

Arbitrageurs wager in the second period. Unlike noise traders, arbitrageurs know p_i, and they do not receive direct utility from gambling. Let x_i be the amount wagered by arbitrageurs on outcome i. Normalizing total wagering by noise traders to 1, the expected return on a \$1 wager on outcome i is:

$$\mathbb{E}[V(\$1 \text{ on } i)] = p_i(1-\sigma)\frac{1+\sum_j x_j}{s_i^* + x_i}$$

The track price discriminates by offering arbitrageurs a rebate, r, on each dollar wagered regardless of the outcome.[29] For every dollar wagered by noise traders, the track receives:

$$\pi(r) = \sigma + (\sigma - r)\sum_{i=1}^{N} x_i^*$$

We assume 1) arbitrage is competitive, and 2) the track optimizes. Hence, the second-period equilibrium is defined by arbitrage volumes $\{x_i^*\} \geq 0$ and a rebate r^* such that:

1. Arbitrageurs make zero profits: $\mathbb{E}[V(\$1 \text{ on } i)] = \$1 - r^*$ when $x_i^* > 0$

2. Arbitrageurs leave no money on the table: $\mathbb{E}[V(\$1 \text{ on } i)] \leq \$1 - r^*$ when $x_i^* = 0$

3. The track chooses the income-maximizing rebate: $r^* = \arg\max_r \pi(r)$

[29] In practice, ADWs set the rebates. We describe tracks and ADWs as acting jointly, given that their (short-run) incentives are aligned—i.e., to maximize income from arbitrageurs.

6.2 Results

In the first-period Walrasian equilibrium, $s_i^* = q_i$, and only those for whom $u_j > \sigma$ invest. First-period parimutuel odds are $O_i^{(1)} = (1-\sigma)/q_i - 1$.

For the second-period, we first derive the condition under which an equilibrium exists.

Lemma 1. *Reorder the outcome indices i such that:*

$$\frac{p_1}{q_1} \geq \frac{p_2}{q_2} \geq \cdots \geq \frac{p_i}{q_i} \geq \frac{p_{i+1}}{q_{i+1}} \geq \cdots \geq \frac{p_N}{q_N}$$

An equilibrium exists if and only if there exists an index m such that:

$$\frac{p_m}{q_m} > \sqrt{\frac{P_m(1-P_m)}{Q_m(1-Q_m)}} \geq \frac{p_{m+1}}{q_{m+1}}, \tag{2}$$

where $P_m \equiv \sum_{i=1}^{m} p_i$ and $Q_m \equiv \sum_{i=1}^{m} q_i$. In this equilibrium, $x_i > 0$ for $i \leq m$ and $x_i = 0$ for $i > m$.

Proof. In Appendix C.1.

The index m separates outcomes that arbitrageurs wager on from those they do not. P_m and Q_m are the cumulative subjective probabilities—held by arbitrageurs and noise traders, respectively—among outcomes wagered on by arbitrageurs.

Arbitrageurs wager on outcomes for which their own beliefs are high relative to those of noise traders:

Corollary 1. $P_m \geq Q_m$.

Proof. In Appendix C.2.

Under a mild condition, an equilibrium exists:

Proposition 1. *If $p_1 > q_1$, then there exists an $m \in \{1, \ldots, N-1\}$.*

Proof. In Appendix C.3.

An equilibrium exists so long as subjective beliefs do not coincide for every outcome—i.e., if $p_1 > q_1$ (and hence, $p_N < q_N$). If so, $m \geq 1$, implying that arbitrageurs always wager on the outcome with the highest ratio of subjective beliefs. In addition, $m < N$, implying that arbitrageurs never wager on the outcome with the lowest ratio of subjective beliefs.

Equilibrium wagers by arbitrageurs are:

$$x_i^* = p_i\sqrt{\frac{Q_m(1-Q_m)}{P_m(1-P_m)}} - q_i \text{ for } i \in \{1, \ldots, m\} \tag{3}$$

and $x_i^* = 0$ for $i \in \{m+1, \ldots, N\}$. The amount wagered by arbitrageurs on outcome i is increasing in its true probability, p_i, and decreasing in the beliefs of noise traders, q_i, all else equal.

Other quantities all have analogous expressions to those in Section 3, with P_m replacing p and Q_m replacing q. Let:

$$\gamma_m \equiv \frac{P_m(1-Q_m)}{Q_m(1-P_m)}$$

Arbitrageurs collectively wager:

$$\sum_{i=1}^{N} x_i^* = Q_m(\sqrt{\gamma_m} - 1) \tag{4}$$

Let r^* be the revenue-maximizing rebate. Then,

$$\sigma - r^* = (1-\sigma)(1-P_m)(\sqrt{\gamma_m} - 1) \tag{5}$$

The final parimutuel odds are:

$$O_i^{(2)} = (1-\sigma)\begin{cases} \frac{1}{p_i}\left[P_m + (1-P_m)\sqrt{\gamma_m}\right] - 1, & i \in \{1, \ldots, m\} \\ \frac{1}{q_i}\left[Q_m\sqrt{\gamma_m} + (1-Q_m)\right] - 1, & i \in \{m+1, \ldots, N\} \end{cases} \tag{6}$$

Before the rebate, the true expected value of a \$1 wager on outcome i is:

$$\mathbb{E}[V(\$1 \text{ on } i)] = (1-\sigma)\begin{cases} P_m + (1-P_m)\sqrt{\gamma_m}, & i \in \{1, \ldots, m\} \\ \frac{p_i}{q_i}\left[Q_m\sqrt{\gamma_m} + (1-Q_m)\right], & i \in \{m+1, \ldots, N\} \end{cases} \tag{7}$$

Arbitrage equalizes returns among outcomes on which arbitrageurs wager. A \$1 wager on any of the first m outcomes has an expected return of $\$1 - r^*$. For $i > m$, wagers perform worse in expectation, following from (2), and they perform progressively worse as i increases, given that p_i/q_i decreases in i.

It is straightforward to see how a favorite-longshot bias in the track's predictions manifests in the parimutuel odds. A favorite-longshot bias characterizes the track's predictions if $p_1 \geq p_2 \geq \cdots \geq p_N$, as this implies that the track underestimates the favorite ($p_1 > q_1$) and overestimates the longshot ($p_N < q_N$). Further assume that $q_1 \geq q_2 \geq \cdots \geq q_N$—i.e., the track's predictions correctly order the outcomes by their true probabilities. Given that the parimutuel odds in (6) are inversely proportional to p_i, this indexing orders outcomes by their odds, from short to long. For outcomes with shorter odds (i.e., $i \leq m$), expected returns are flat. For outcomes with longer odds (i.e., $i > m$), expected returns are decreasing. Hence, the relationship between odds and expected returns is piecewise linear, similar to that observed in Figure 2, with the kink located at the odds of outcome m.

6.3 Estimation

We infer beliefs from the morning lines. Noise traders take the morning-line odds at face value as in (1). In contrast, we imbue arbitrageurs with the well-calibrated beliefs, $p_{i,\text{win}}$, shown in Figure 7.[30] Noise traders assume that the morning lines imply well-calibrated beliefs. Arbitrageurs ensure that their beliefs are well calibrated.

We construct beliefs for exotic wagers from beliefs in the win pool using the Harville formula. Assuming that outcomes are independent, the probability of finishing in n^{th} place is the probability of winning a race without the first $n-1$ finishers (Harville, 1973). Appendix D lists transformations for each pool.

Finding the equilibrium in each pool proceeds according to the model. We order outcomes by p_i/q_i, from greatest to smallest. Using a grid search, we find the index m that satisfies the relationship in (2); if more than one such index exists, we choose the one that maximizes track income.[31,32]

[30]Because this routine is computationally intensive, we do not repeat it during bootstrapping. In other words, we treat p_i as data rather an estimate.

[31]The expressions for equilibrium quantities are slightly different in place and show pools, where more than one outcome obtains. Let k denote the number of outcomes that pay out—i.e., $k = 2$ in place pools and $k = 3$ in show pools—with $\sum_i q_i = \sum_i p_i = k$. Observe that the parimutuel pays out $1/k^{th}$ of the post-tax pot to winning wagers. Hence, second-period odds can be written as:

$$O_i^{(2)} = \frac{1-\sigma}{k} \frac{1+\sum x}{q_i/k + x_i} - 1,$$

Modified expressions for the equilibrium odds, expected returns, optimal rebate, and track revenues can be derived in the same manner.

[32]We also restrict the set of possible equilibria to those with non-negative rebates. The model predicts zero rebates in just 3% of race-pools.

6.4 Example

Table 2 illustrates the estimation routine for an example race at Charles Town. T Rex Express, the favorite with 1/1 morning-line odds, finished with final odds of 3/10 on the tote board and in first place on the track. As a result, win bets on T Rex Express paid out a divided of 30 cents on every dollar wagered. Relative to the morning lines, parimutuel odds lengthened for the other six horses in the race.

Table 2: Example race at Charles Town.

	Odds		Beliefs			Model predictions in win pool				
Name	M/L	Final	q_i	p_i	p_i/q_i	$O_i^{(1)}$	$\mathbb{E}_i^{(1)}$	x_i^*	$O_i^{(2)}$	$\mathbb{E}_i^{(2)}$
T Rex Express	1/1	0.3	.41	.55	1.34	1.0	1.11	0.3	0.6	0.89
Tribal Heat	2/1	4.0	.27	.27	0.98	2.0	0.81	0.05	2.3	0.89
Click and Roll	8/1	15.6	.09	.06	0.63	8.0	0.52	0	10.8	0.68
Boston Banshee	30/1	49.7	.03	.02	0.59	30.2	0.49	0	39.5	0.63
Dandy Candy	6/1	6.9	.12	.07	0.58	6.0	0.48	0	8.2	0.63
Eveatetheapple	20/1	69.3	.04	.02	0.50	20.1	0.41	0	26.5	0.54
Movie Starlet	20/1	25.5	.04	.02	0.50	20.1	0.41	0	26.5	0.54

The beliefs ascribed to noise traders, q_i, are those of a risk-neutral bettor who believes that the final odds will coincide with the morning-line odds. Specifically, they are inversely proportional to the returns implied by the morning-line odds, up to a normalizing constant, as in (1). By contrast, the beliefs ascribed to arbitrageurs, p_i, are well calibrated—i.e., proportional to the track-specific rates at which horses with implied beliefs q_i actually win, as shown in Figure 7. At Charles Town, the oddsmaker tends to underestimate the chances of favorites and overestimate the chances of longshots. In this race, $p > q$ for T Rex Express ($q = 0.41$), $p = q$ for Tribal Heat ($q = 0.27$), and $p < q$ for the other 4 horses.

The first-period odds in the win pool, $O_i^{(1)}$, reflect wagering by risk-neutral noise traders, given beliefs q_i. They diverge from the morning-line odds only because the morning-line odds do not precisely reflect the track's take. The first-period odds generate a severe favorite-longshot bias. The first-period expected returns on a \$1 wager—denoted $\mathbb{E}_i^{(1)}$ and taken over p_i—are sharply decreasing in the odds. At first-period odds of 1.0, the $p = 0.55$ favorite is better than an even money bet. By contrast, at odds of 20.1, the two horses with 20/1 morning lines return just 41 cents on every dollar wagered.

The second-period equilibrium consists of a set of wagers by arbitrageurs, x_i^*, such that they make zero profits and leave zero profits on the table. To find this equilibrium, we sort the outcomes in descending order of p_i/q_i and find the equilibrium index m. In the example above, this ordering roughly sorts the horses in ascending order of the morning-line odds—a product of the favorite-longshot bias embedded in the morning lines. The income-maximizing equilibrium is $m = 2$, implying that $x_i > 0$ for the first 2 horses and $x_i = 0$ for the remaining 3. Second-period wagers concentrate on the favorite, and the predicted odds, $O_i^{(2)}$, shorten for the first two horses and lengthen for all others. A less severe favorite-longshot bias remains. For any of the first three horses, the expected return on a \$1 wager, $\mathbb{E}_i^{(2)}$, is 89 cents, with the 11-cent expected loss equaling the optimal rebate. For the longshots, expected losses attenuate in the second period. A \$1 wager on either of the horses with 20/1 morning lines, for instance, returns 54 cents in expectation at odds of 26.5, or 13 cents more than at first-period odds of 20.1.

In the win pool, arbitrageurs wager 35 cents for every dollar wagered by the noise traders. However, income from arbitrageurs comprises just 10% of the track's total income in the win pool, as the track offers arbitrageurs an 11-cent rebate on the 17.25-cent takeout. Arbitrage accounts for larger shares of the track's income in exotic pools. In the exacta pool, for instance, that share is 14%. The compound beliefs attached to outcomes in exotic pools amplify differences in beliefs about win probabilities between noise traders and arbitrageurs and hence, the size of the arbitrage opportunity.

7 Model predictions

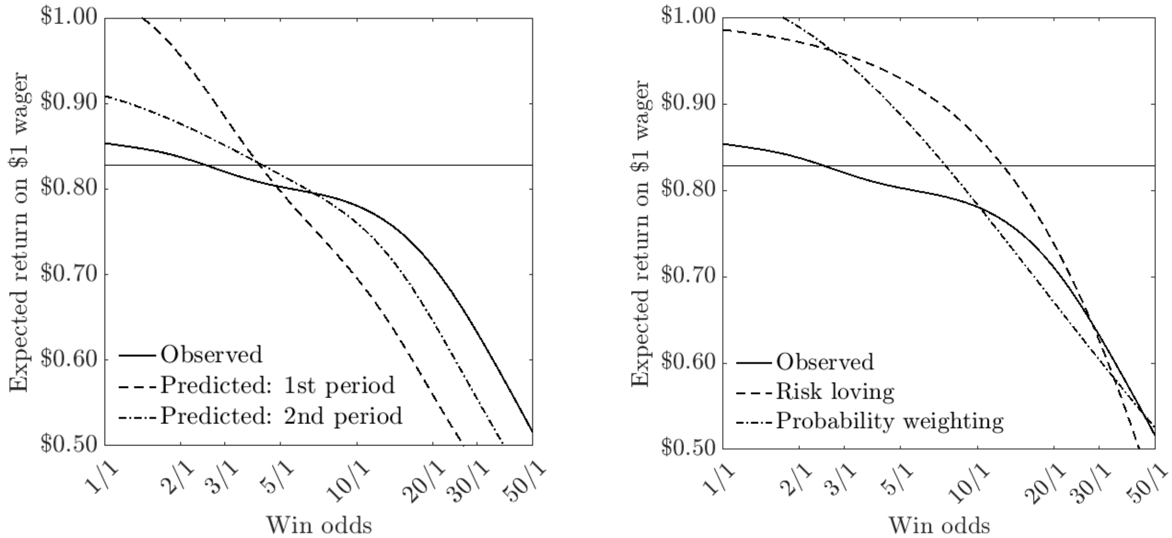
We begin by comparing predictions from the first- and second-period equilibria. Figure 9a shows a smoothed estimate of the relationship between log predicted odds and predicted returns for win bets, separately by period. (All predicted returns reflect expected returns absent the rebate and under well-calibrated beliefs.) In the first-period equilibrium, predicted odds approximate the morning-line odds, which embed a favorite-longshot bias that is more severe than that observed in the data.

The inclusion of arbitrageurs in the second period moderates the bias, and the resulting predictions more closely follow observed returns. For a 1/1 favorite, first-period expected returns exceed observed returns by 16 cents; after arbitrage, the difference is 5.5 cents. For a 30/1 longshot, observed returns exceed first-period expected returns by 15 cents; after arbitrage, the difference is 7.8 cents.

We quantify the goodness of our model's predictions by measuring the average deviation: the mean absolute distance between predicted and observed returns, with the expectation

Figure 9: Predicted and observed returns for win bets. The horizontal line marks the average takeout.

(a) Our model **(b)** Representative-agent models



Note: Estimated using local linear regression with a Gaussian kernel. For comparability, we use the same bandwidth for the predictions as for the observed estimate, of 0.50 log odds.

taken over the distribution of observed odds.[33] For a bettor who wagers randomly, the mean difference between her returns and those predicted by the model converges asymptotically to our average deviation measure. The average deviation is 3.8 (se: 0.6) cents for the model's second-period predictions, compared to 9.1 (0.5) cents before arbitrage.

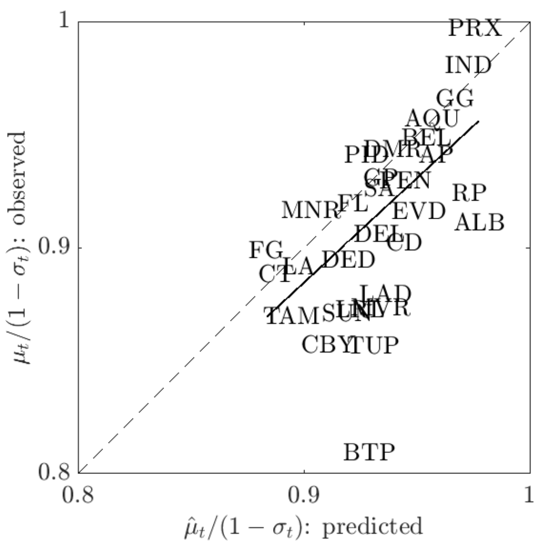
Representative-agent models perform worse. Previous work has rationalized the favorite-longshot bias with risk-loving preferences (e.g., Weitzman, 1965; Ali, 1977) or a tendency to overweight small probabilities (e.g., Snowberg and Wolfers, 2010). Figure 9b shows predicted returns from each model.[34] For both models, predicted returns for favorites exceed observed

[33] We estimate this distribution using a kernel density estimator with a Gaussian kernel and Silverman's rule-of-thumb bandwidth.

[34] We estimate these models following Snowberg and Wolfers (2010). The equilibrium conditions are $p_i U(O_i + 1) = 1 \; \forall i \in \mathcal{R}$ for the risk-loving model and $\pi(p_i)(O_i + 1) = 1 \; \forall i \in \mathcal{R}$ for the probability-weighting model. Each model is governed by one parameter. For the risk-loving model, Snowberg and Wolfers (2010) use the CARA utility function $U(x) = \big(1 - \exp(-\alpha x)\big)/\alpha$, where α modulates the agent's risk tolerance. For the probability-weighting model, they use the weighting function $\pi(p) = \exp\big[-\big(-\log(p)\big)^{\beta}\big]$, where β modulates the degree to which the agent overweights small probabilities and underweights large ones (Prelec, 1998). We estimate each model by minimizing the squared distance between observed returns and expected returns, $p_i(O_i + 1)$, where O_i are the observed parimutuel odds, and p_i can be found by solving the appropriate equilibrium condition. In the risk-loving model, we estimate $\hat{\alpha} = -0.029$ (0.001), implying an

returns, and the overall fit is poor. Average deviations, of 9.2 (0.5) cents for the risk-loving model and 7.0 (0.4) cents for the probability-weighting model, exceed the 3.8-cent deviation from our model with arbitrage. The difficulty is that neither model can rationalize highly negative returns for probable events. Risk-loving agents pay a large premium for lottery tickets, but they pay a small premium for gambles with little upside. Prospect-theory agents overweight large probabilities and thus need to *receive* a premium in order to wager on likely events. Neither explanation can account for a willingness to lose considerable sums, in expectation, when wagering on favorites.[35]

Figure 10: Predicted and observed mean returns, by track.



Our model with arbitrage also captures differences across tracks in the extent of the bias. Figure A2 in Appendix A shows estimates of the observed and predicted relationships between log odds and expected returns, separately by track. For most tracks, the predicted relationship follows the observed relationship. Figure 10 summarizes the model's fit at each

extreme taste for risk. This representative agent is indifferent between \$100 for sure and a gamble that pays \$123 with 50% probability and \$0 otherwise. In the probability-weighting model, we estimate $\hat{\beta} = 0.900$ (0.003), implying overweighting of small probabilities. This representative agent behaves as if an event with 1.0% probability occurs 1.9% of the time. These estimates are more slightly more extreme than those of Snowberg and Wolfers (2010), who estimate $\hat{\alpha} = -0.017$ and $\hat{\beta} = 0.928$. Snowberg and Wolfers (2010) use these estimates to predict the odds for exotic bets, which they compare to the observed odds in the Jockey Club database. They find that both models have large prediction errors, though the errors are smaller for the probability-weighting model. Unfortunately, we cannot replicate this analysis using our data, as odds for exotic bets are not listed on the race chart.

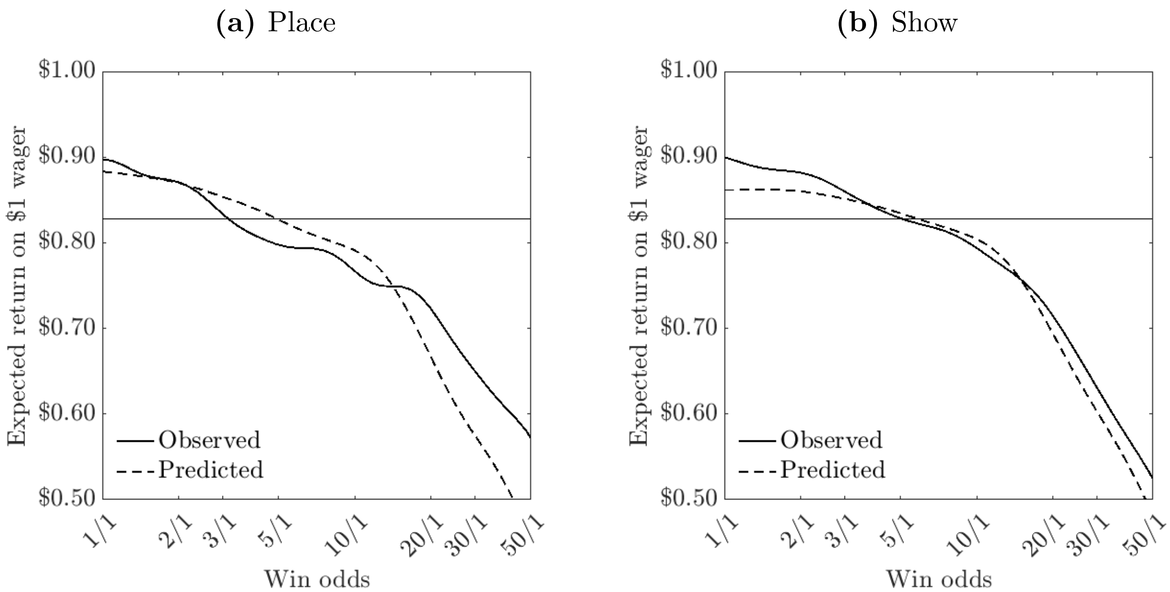
[35] These explanations provided a better fit decades ago, when favorites were close to even-money bets (Thaler and Ziemba, 1988). See the end of Section 7.2 for a discussion of why the favorite-longshot bias has moderated over time.

track by comparing observed and predicted mean returns—i.e., the expected return on a randomly placed \$1 win bet, normalized by the state-sanctioned return, $1 - \sigma_t$. Observed and predicted values are correlated at 0.59.

7.1 Other pools

We evaluate our second-period predictions in other pools as well. Since the race chart only lists final odds in the win pool, in other pools, we estimate the relationship between log odds in the *win* pool and expected returns in the given pool.

Figure 11: Observed and predicted returns for place and show bets. The horizontal line marks the average takeout.



Note: Estimated using a local linear regression with a Gaussian kernel. The bandwidth, of 0.21 log odds for place bets and 0.26 log odds for show bets, minimizes the leave-one-out mean-squared error in the observed data.

Figure 11 shows observed and predicted returns in place (11a) and show (11b) pools. A favorite-longshot bias characterizes returns for both place and show bets. In both pools, favorites with 1/1 win odds return 90 cents on the dollar, and longshots with 30/1 win odds return about 65 cents on the dollar. The model fit is close in both pools, with an average deviation of 3.4 cents for place bets and 1.6 cents for show bets.

The final three sets of figures show expected returns on wagers involving two horses—those in exacta (12), quinella (13), and daily-double (14) pools. In each set of pool-specific

Figure 12: Exacta: expected return on $1 wager.

(a) Observed **(b)** Predicted **(c)** Observed − predicted

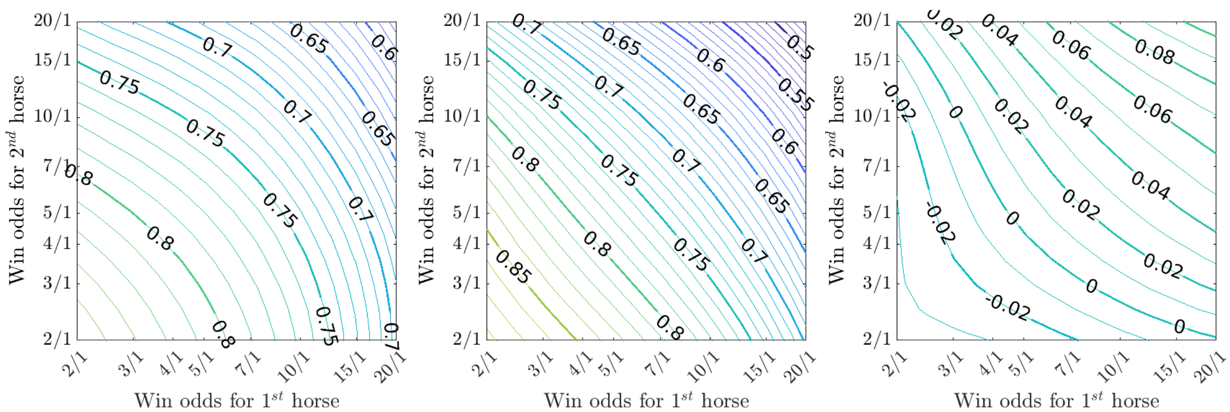


Figure 13: Quinella: expected return on $1 wager.

(a) Observed **(b)** Predicted **(c)** Observed − predicted

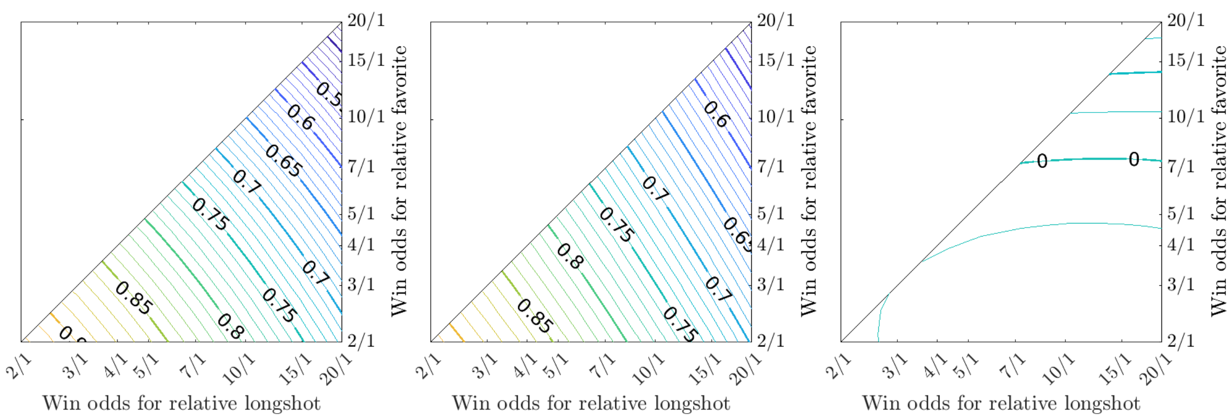
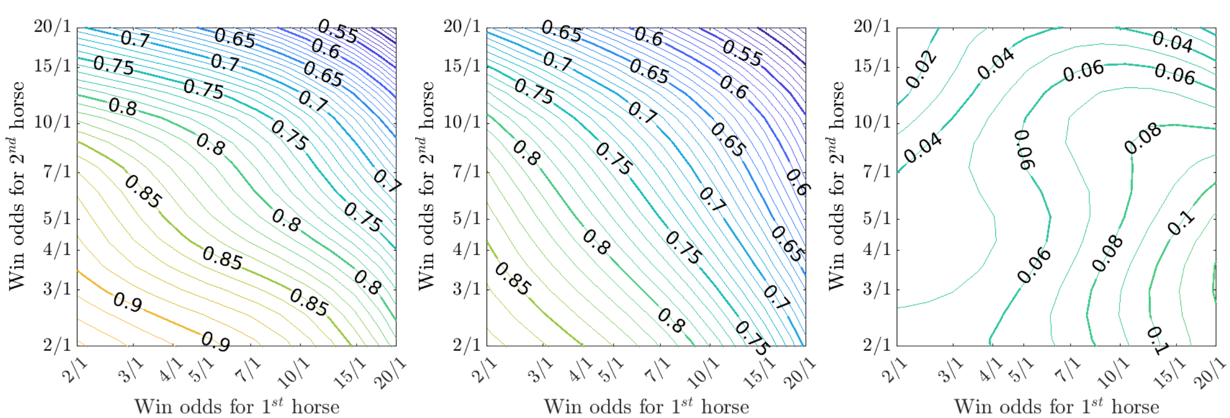


Figure 14: Daily double: expected return on $1 wager.

(a) Observed **(b)** Predicted **(c)** Observed − predicted



figures, subfigure (a) shows observed returns, subfigure (b) shows predicted returns, and subfigure (c) shows the difference between observed and predicted returns. In all figures, expected returns are shown as a function of log win odds for the first and second horses chosen. (Since finishing order is irrelevant for quinella wagers, we show expected returns as a function of log win odds for the relative longshot and relative favorite.)[36]

For exotic bets, the favorite-longshot bias is severe. In expectation, wagering $1 on two randomly selected horses returns 67 cents in exacta pools, 65 cents in quinella pools, and 68 cents in daily-double pools—incurring far larger losses than the average takeouts of 21, 22, and 20 cents, respectively. Predicted returns from our model approximate observed returns in each pool, with average deviations of 5.7 cents for exacta bets, 1.5 cents for quinella bets, and 6.4 cents for daily-double bets.

7.2 Income from arbitrage

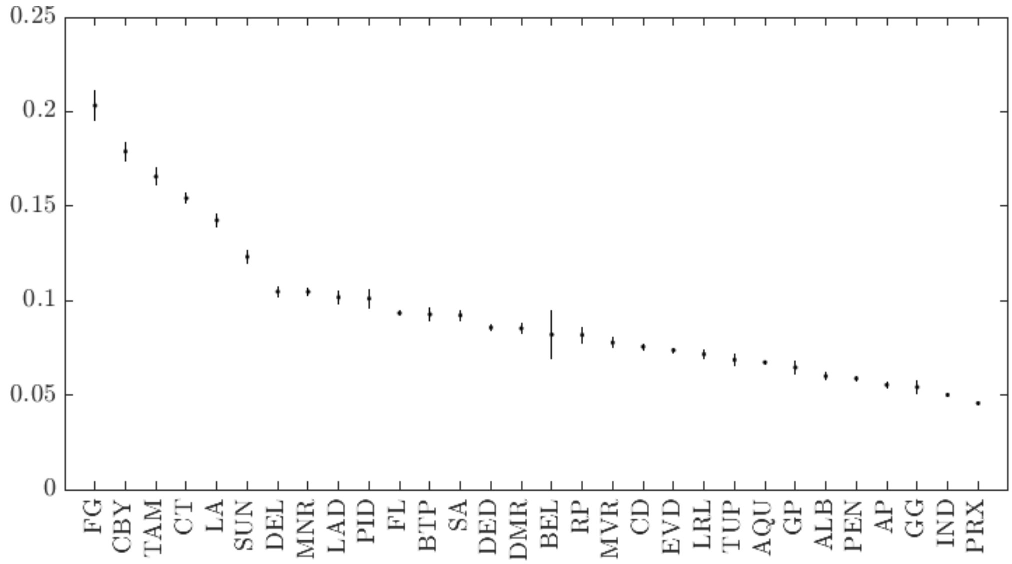
Our model also predicts the track's income from noise traders and arbitrageurs. Figure 15 shows the share of gambling income from arbitrage, separately by track.[37] We estimate considerable variation in the extent to which tracks profit from arbitrage. Those that promulgate well-calibrated morning-line odds (e.g., PRX) generate minimal income from arbitrage. At other tracks, arbitrage accounts for as much as 20% of gambling income—or equivalently, a 25% increase on the track's income from noise traders.

The plots in Figure A3 in Appendix A show these shares separately by track and pool; the plots also show the takeout rates paid by noise traders and the average takeout rates paid by arbitrageurs after the rebate. Across tracks, arbitrage volumes tend to be higher in exotic pools. This occurs because exotic wagers are tied to compound outcomes, which amplify differences in beliefs between noise traders and arbitrageurs.

[36]Each figure is estimated using a local linear regression with a bivariate Gaussian kernel. In each pool, the bandwidth pair minimizes the leave-one-out mean-squared error in the observed data (a).

[37]We weight each betting pool by the total amount wagered. For win, place, and show pools, the race chart lists the combined amount wagered rather than the pool-specific amounts. We assume that the listed total was wagered entirely in the win pool, reflecting the unpopularity of place and show bets. At the Santa Anita race track, for example, nearly two in three dollars wagered in the win, place, or show pools is wagered in the win pool (`https://lat.ms/2il84x7`). The model's predictions are generally similar across the three pools, as shown in Figure A3, and the estimates in Figure 15 are meaningfully unchanged when dividing the pot equally among the win, place, and show pools.

Figure 15: Share of track income from arbitrageurs, by track.



7.3 Discussion of results

We estimate a parsimonious model. As a result, it is almost certainly wrong. We assume that all bettors are risk-neutral. Surely, many bettors are not. We assume that no bettor suffers from cognitive biases such as probability weighting. Surely, some biases plague at least some bettors. We assume that arbitrage is competitive. In practice, competition among arbitrageurs may be imperfect. Finally, we assign coarse beliefs to arbitrageurs by estimating win rates from a single variable: the morning-line odds. However, arbitrageurs have access to other information, such as race histories, from which to form more refined beliefs.

Nonetheless, our model closely predicts the extent of the favorite-longshot bias, and it predicts variation in the degree of the bias across tracks. The basic logic captured by our model—noise traders trade on the morning lines, arbitrageurs trade on well-calibrated beliefs, and the track tries to maximize gambling revenue when setting the rebate—provides a good first-order approximation. One reason for this is that the odds predicted by our model are somewhat robust to misspecification. For instance, collusion by arbitrageurs, ceteris paribus, will reduce arbitrage volume, thereby limiting the extent to which arbitrage shortens the odds of favorites and lengthens the odds of longshots. But optimizing tracks will respond to collusion by increasing rebates, which induces arbitrageurs to wager larger sums.

This robustness argument does not hold for our estimates of track income. If noise traders are risk-loving, or if they overweight small probabilities, or if arbitrageurs trade on more refined beliefs than we assign them, then arbitrageurs will tend to disagree more strongly with noise traders, and arbitrage opportunities will be larger. If instead noise traders are risk-averse, if arbitrageurs collude, or if tracks do not set the rebate optimally, then our model will overestimate the track's income from arbitrage.

Nevertheless, our estimates of arbitrage volume and rebates are consistent with those from industry sources. At an industry conference in 2012, the National Horsemen's Benevolent and Protective Association estimated that "just less than 20% of total pari-mutuel handle...come[s] from high-volume shops,"[38] or the Advance Deposit Wagering companies that administer rebates to volume bettors. Using our model, we estimate that arbitrageurs account for 18.1% of wagering volume. At the same conference, one ADW advertised an average rebate for its clients of 16.7% during a five-day period at Beulah Park in Ohio. That track has since closed. However, the owners subsequently opened the Mahoning Valley (MVR) racetrack under the same license, where we estimate an average rebate of 12.9%.

7.4 Additional tests

We relax two assumptions from our model. First, we consider a less flexible rebate structure, in which the track sets the profit-maximizing rebate in each betting pool.[39] Figure A4a in Appendix A shows the relationship between odds and expected returns for win bets; Figure A4b shows a correlation of 0.57 between predicted and observed mean returns. Both plots closely parallel what we observe with a variable rebate.

Second, we assign arbitrageurs more refined beliefs by estimating win rates from race characteristics such as the weather and distance, in addition to the implied probabilities from the morning lines.[40] (Unfortunately, we cannot construct reliable measures of past performance for horses, jockeys, trainers, and owners from our data.) Our estimator tends to predict more extreme probabilities than those predicted by the track, and the inclusion of race variables allows for the degree of that transformation to vary by race. Including race variables improves our out-of-sample predictions of win rates. With these variables,

[38] `https://www.bloodhorse.com/horse-racing/articles/132355`.

[39] When this rebate predicts negative odds in a given race-pool, we set the rebate to 0 for that race-pool.

[40] These variables comprise indicators for whether the race is 1) run on a weekend, 2) a maiden race (in which none of the horses have previously raced), 3) a claiming race (in which the winning horse can be purchased), 4) for thoroughbred horses, 4) run on dirt (at tracks where the surface varies), and 5) whether the track is sealed. We also include ordinal variables describing 1) the condition of the track, and 2) the amount of precipitation. And we include continuous variables for 1) the length of the race, 2) the total prize money, and 3) the share of that prize awarded to the winner. As before, we first decompose the matrix of race-specific variables, and we use the eigenvectors of that matrix as inputs, as described in Appendix B.

our estimator assigns an average probability of 26.9% to race winners, compared to 21.5% without, a 25% improvement.

The model continues to predict a favorite-longshot bias when seeded with higher resolution beliefs, and it still predicts the variation in the extent of that bias across tracks. Figure A5a shows a slightly steeper favorite-longshot bias for win bets with higher resolution beliefs. Figure A5b shows a correlation of 0.59 between predicted and observed mean returns. There is a difference in levels, however. With the lower resolution beliefs, our model on average underestimates mean returns; with the higher resolution beliefs, our model overestimates mean returns.

More refined beliefs inflate our estimates of track income from arbitrage. With higher resolution beliefs, arbitrageurs tend to disagree more strongly with noise traders, amplifying arbitrage opportunities. Rebates are lower, and arbitrage volumes are higher, with arbitrage comprising between 31% and 68% of gambling revenues across tracks.

8 Optimal deception

We return to our stylized model from Section 3 to derive the optimal amount of deception for the broker. In doing so, we answer two open questions. Why do some tracks bias their predictions more than others? And why do tracks underestimate the chances of longshots, rather than the reverse?

Begin with the broker's income under the optimal rebate. For every dollar wagered by noise traders, the broker earns:

$$\pi(q|p) = \sigma + (1-\sigma)\left[1 - \left(\sqrt{pq} + \sqrt{(1-p)(1-q)}\right)^2\right] \tag{8}$$

The first term is the broker's income from noise traders, and the second term is income from arbitrageurs. The broker's income from arbitrage is proportional to one minus the squared Bhattacharyya coefficient, which measures the similarity between two distributions. Hence, the broker's income from arbitrage, and thus total income, is increasing in the divergence between p and q. When the broker tells the truth, the Bhattacharyya coefficient is 1, and $\pi = 0$. When the broker lies, the coefficient is less than 1, and $\pi > 0$.

It is apparent that maximal deception maximizes the broker's income. This may not be advisable in the long run. If the event occurs, noise traders will know they were deceived. Even if it does not, large swings in second-period parimutuel returns will likely arise suspicion. In response, noise traders may seek out predictions from other sources, decide to invest with other brokers, quit investing altogether, or choose to become sophisticated—all of which decrease brokerage fees in the future. Less aggressive deception makes statistical inference more difficult and dampens swings in parimutuel returns, and is presumably less costly for the broker.

This logic implies that different time horizons will yield different choices of q. Brokers who expect to be in business for decades will fear losing customers, whereas those whose future is in doubt will be less afraid to lie to their patrons. While there are too few tracks in North America—let alone in our data—to test this hypothesis empirically, one anecdote provides support. Despite lowering their takeout below state-regulated thresholds, Canterbury Park (CBY) saw on-track wagering decrease from 2015 to 2016.[41] At the same time, Canterbury Park aggressively overestimated the chances of longshots, as seen in Figure 7.[42] A track with worsening finances—and hence, a strong reason to discount the future—is among the most misleading.

To capture these dynamics, we subtract a cost from the broker's income that is increasing in the divergence between p and q—i.e., in the degree of deception. In particular, we assume that the cost of deception, $C_\alpha(q|p)$, is proportional to the Rényi (or α) divergence between p and q:

$$C_\alpha(q|p) \propto \frac{1}{\alpha - 1} \log \left(\frac{p^\alpha}{q^{\alpha-1}} + \frac{(1-p)^\alpha}{(1-q)^{\alpha-1}} \right),$$

The Rényi divergence has a single parameter, $\alpha \geq 0$, which modulates the function's convexity.[43] Its limiting case as $\alpha \to 1$ is the Kullback-Leibler divergence: $p \log[p/q] + (1 - p) \log[(1-p)/(1-q)]$. Figure 16a shows Rényi divergences for different values of α. Regardless of α, the divergence is zero when $q = p$, positive when $q \neq p$ (strictly so when $\alpha > 0$), and increasing as q separates from p. For any α, the function is globally convex in q (Van Erven and Harremos, 2014). At $q = p$, the convexity in q is proportional to α.

Figure 16b shows the broker's income, cost, and profit when the cost of deception is proportional to the Kullback-Leibler divergence. Both income and cost increase with distance between p and q. When q is close to p, income increases faster than cost, implying that deception is profitable. In general, deception is profitable when the income function is more

[41] `http://www.thoroughbreddailynews.com/oped-why-canterburys-takeout-reduction-didnt-work/`

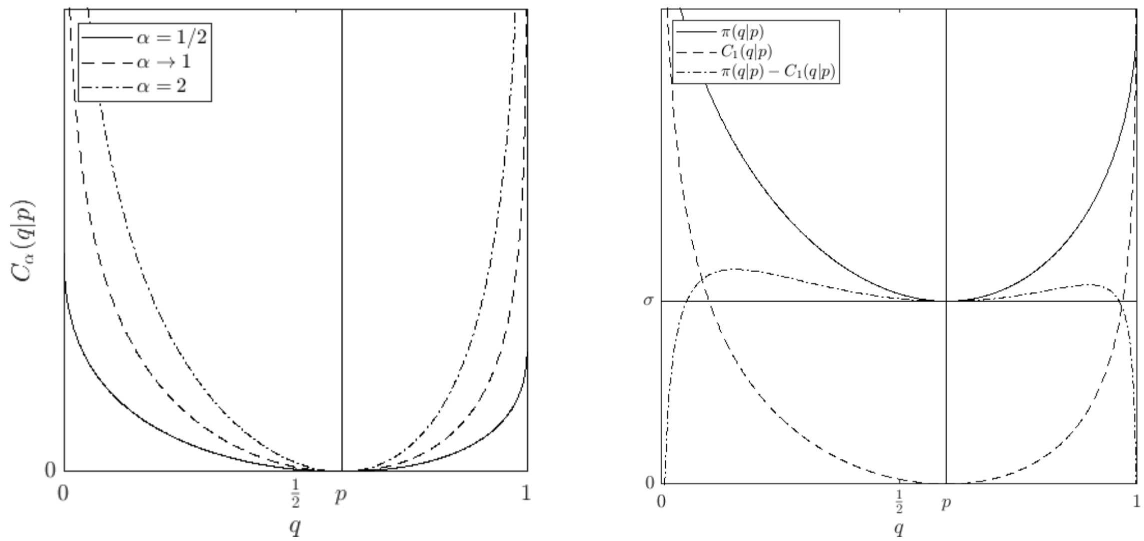
[42] At Canterbury, horses with 15-1 morning lines (average q of 5.6%) won 2.8% of races, horses with 20-1 morning lines (average q of 4.2%) won 1.1% of races, and horses with 30-1 morning lines (average q of 2.9%) did not win a race in 97 tries.

[43] Incidentally, the Rényi divergence is also commonly described as the most important divergence in information theory (e.g., Van Erven and Harremos, 2014).

Figure 16: Rényi divergence between q and $p = 0.6$ for different values of α (a), and broker financials when the cost of deception is proportional to the Kullback-Leibler divergence, as $\alpha \to 1$ (b).

(a) Rényi divergence

(b) Broker income, cost & profit



Note: In (b), $\sigma = 0.2$ and $C_1(q|p)$ includes a multiplicative constant of $(1-\sigma)/3$.

convex at $q = p$ than the cost function.[44]

When deception is profitable, the profit function has two local maxima—one when the broker underestimates the favorite and one when the broker overestimates the favorite. Which is preferable? If $\alpha = \frac{1}{2}$, the broker is indifferent between underestimation and overestimation.

Lemma 2. *If* $\alpha = \frac{1}{2}$, *then* $\max_{q \geq p} \pi(q|p) - C_\alpha(q|p) = \max_{q \leq p} \pi(q|p) - C_\alpha(q|p)$.

Proof. In Appendix C.4.

Using this lemma, we show that the broker overestimates the favorite when $\alpha < \frac{1}{2}$ and underestimates the favorite when $\alpha > \frac{1}{2}$.

Proposition 2. *Let* $q^* = \arg\max_q \pi(q|p) - C_\alpha(q|p)$. *If* $\alpha < \frac{1}{2}$, *then* $q^* \geq p$. *If* $\alpha > \frac{1}{2}$, *then* $q^* \leq p$.

Proof. In Appendix C.5.

[44]This implies an upper bound on the coefficient of proportionality of $(1-\sigma)/2\alpha$.

The broker induces a favorite-longshot bias when $\alpha > \frac{1}{2}$—i.e., when deception has sufficiently convex costs, or when large lies prompt customers to flee quickly and en masse.[45]

More prosaic reasons may also underlie the tracks' overestimation of longshots, for instance, by making the race appear to be more competitive.

9 Discussion

This paper shows that the favorite-longshot bias in horserace parimutuel betting markets is consistent with a parallax-and-tax scheme and requires no primitive biases other than credulity. We write down a model in which the track deceives noise traders so as to profit by taxing arbitrageurs. In the data, the morning-line odds facilitate this deception. Billed as the track's prediction of the final parimutuel odds, the morning lines at most tracks instead embed a favorite-longshot bias. In our model, noise traders take the track's predictions at face value and overbet longshots, thereby making favorites attractive to informed arbitrageurs. Just before the race begins, arbitrageurs bet on favorites, transforming excess losses sustained by noise traders into additional income for the track.

Our account does not rule out non-standard preferences or biases in belief formation, which casual empiricism suggests are widespread at the track. Instead, our model offers a mechanism for how these behaviors can occur at similar rates at tracks that exhibit a favorite-longshot bias and at those that do not. That mechanism is price discrimination: the bias in the final parimutuel odds depends only on the taxes paid by arbitrageurs. In our model, tracks that tell greater lies to noise traders tax arbitrageurs at higher rates, thereby limiting their capacity to correct the bias.

Price discrimination also provide an explanation for two additional stylized facts. First, the favorite-longshot bias has moderated over time. In the 1980s, expected returns on a 1/1 favorite and a 50/1 longshot differed by 50 cents on the dollar (Thaler and Ziemba, 1988); they now differ by 35 cents. Rebates, which emerged with online wagering, increase arbitrage volume, which moderates the bias. Second, the end-of-day effect, in which the favorite-longshot bias is steeper for the last race of the day (Thaler and Ziemba, 1988), has disappeared. Snowberg and Wolfers (2010) do not find this pattern, nor do we. Noise traders may overbet longshots more severely at the end of the day. But if rebates do not account

[45] In financial markets, overpricing has been observed both for positively skewed investments (i.e., longshots), like penny stocks (Kumar, 2009), and for negatively skewed investments (i.e., favorites), like mortgage-backed securities prior to the financial crisis. Our model offers a unifying explanation: the direction of the bias depends on how fast the broker's long-run costs increase in its deception.

for this tendency, then an end-of-day effect will not appear in the final parimutuel odds.

As with most work in forensic economics (for a review, see Zitzewitz, 2012), our evidence points to malfeasance but not necessarily to intentionality. In particular, we cannot say why most tracks promulgate miscalibrated predictions. Some oddsmakers admit that the morning-line odds are intentionally biased, describing an "unwritten rule never to quote odds larger than 30 to 1," or an unwillingness to "point the finger too clearly at a horse's winning chances."[46] Others offer different motives, such as an aversion to extreme errors. "I take a somewhat conservative approach with my morning line," writes the oddsmaker at Arlington Park. "No odds-maker wants to see one of their 6-1 shots sent off as an 8-5 favorite, their 2-1 favorite sent off at 9-2, or a 30-1 morning line outsider at 5-1."[47] Whether or not such motives are true, a parallax-and-tax scheme makes them appear more credible. When a trader manipulates the price of an asset she holds, the motive is obvious. When a broker instead launders profits through taxes on arbitrage, alibis are more persuasive.

Parallax-and-tax schemes appear to characterize trade in other markets. In the lead-up to the financial crisis, investment banks and credit-rating agencies underestimated the risk in mortgage-backed securities. Unsuspecting investors overpriced these securities, generating demand from savvy investors for ways to short them, which the same banks engineered and sold for large fees (Lewis, 2015). As described in a Senate report on the financial crisis, "Goldman [Sachs] marketed Abacus securities to its clients, knowing the CDO was designed to lose value," and then "allowed a hedge fund, Paulson & Co. Inc., that planned on shorting the CDO to play a major but hidden role in selecting its assets...without disclosing the hedge fund's asset selection role or investment objective to potential investors" (Levin et al., 2011, 10). Investment banks created arbitrage opportunities by underestimating risk, and they profited by taxing the arbitrageurs.

Our results also inform a policy debate about gambling, one that has seen renewed interest following the Supreme Court's 2018 ruling that allows states to regulate sports betting. Gambling is commonly rationalized by way of prospect theory. Wagers that lose money on average are appealing to agents who are risk-loving in the domain of losses or who overweight small probabilities (Kahneman and Tversky, 1979; Barberis, 2012). Both of these explanations imply that gambling is a mistake. The gambler will later regret the wagers she placed with the hope of getting back to even, or those she placed with a biased perception of the odds. Under this logic, welfare-enhancing interventions target gamblers, for instance by

[46] Quotes from Snyder (1978, 1117).

[47] `https://web.archive.org/web/20161028154509/www.arlingtonpark.com/node/10222`.

"nudging" them to gamble less. Our results suggest that gambling is better rationalized by simple enjoyment (Conlisk, 1993).[48] In our model, noise traders happily incur losses equal to the state-regulated tax. Presumably, they would be unhappy to learn that they are being misled, and that they incur additional losses as a result. Under this logic, welfare-enhancing regulations target misinformation by brokers.

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[48]There are other reasons that people gamble, such as addiction, that call for different policy responses.

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A Additional figures

Figure A1: Replication of Figures 3b and 8 with weighted mean returns, as described in Footnote 26.

(a) Weighted mean returns

(b) Miscalibration

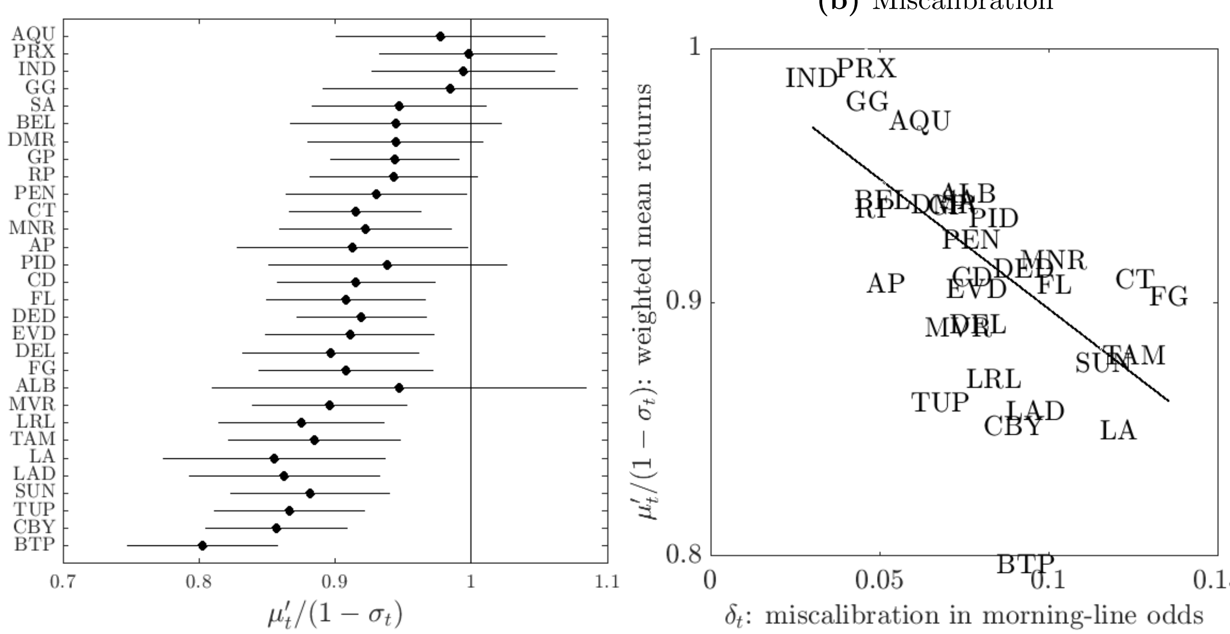
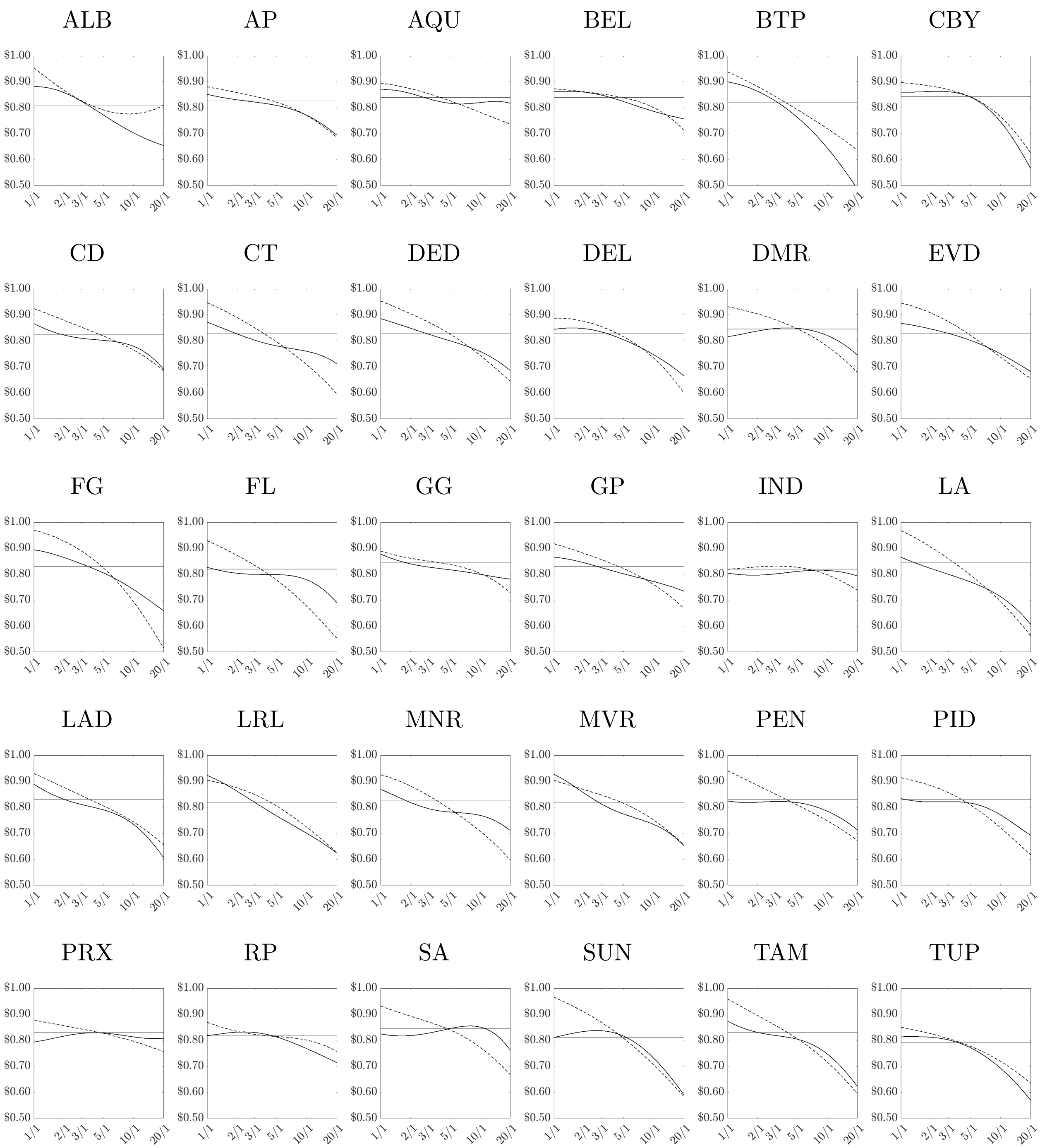


Figure A2: Observed (solid line) and predicted (dashed line) returns for win bets with 95% confidence intervals, by track. The horizontal line marks $1 - \sigma_t$.



Note: Estimated using a local linear regression with a Gaussian kernel. For comparability, we use a common bandwidth for all estimates, of 1.0 log odds.

Figure A3: Share of gambling income from arbitrage (line), the takeout (top of error bar), and the average takeout paid by arbitrageurs (bottom of error bar), separately by track and pool.

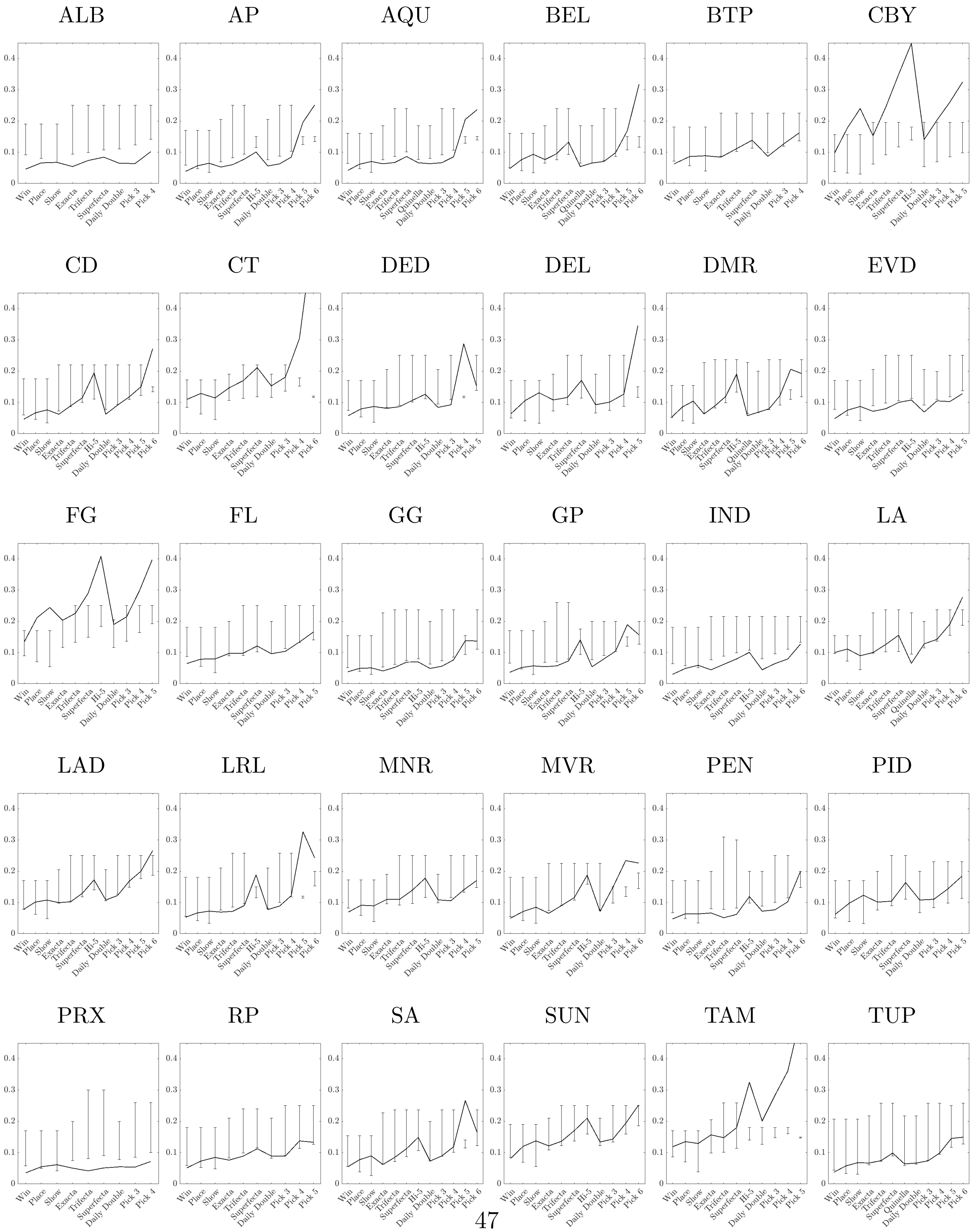
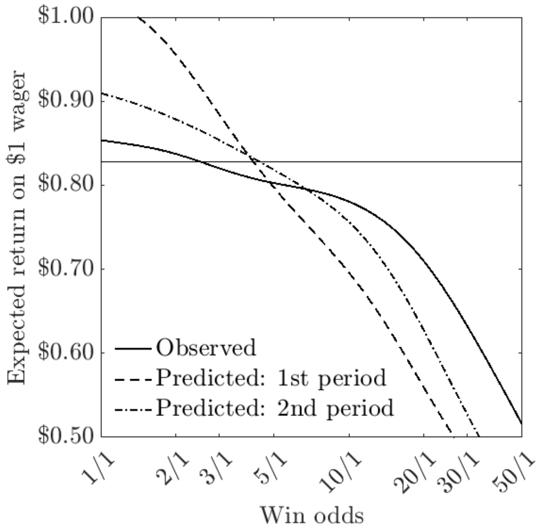


Figure A4: Replications of Figures 9a (a) and 10 (b) when the rebate is fixed within pool.

(a) Odds and expected returns



(b) Mean returns

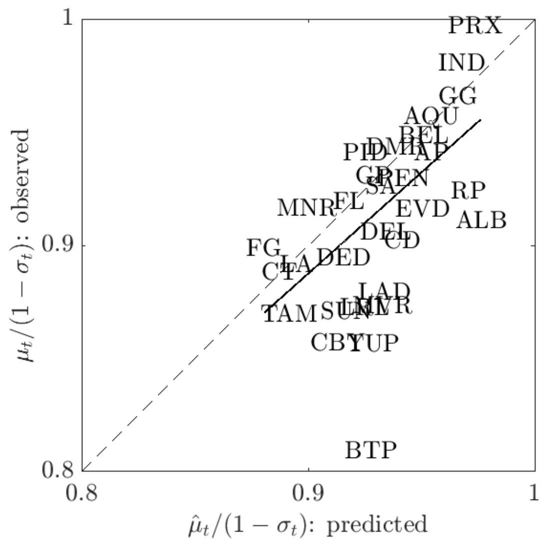
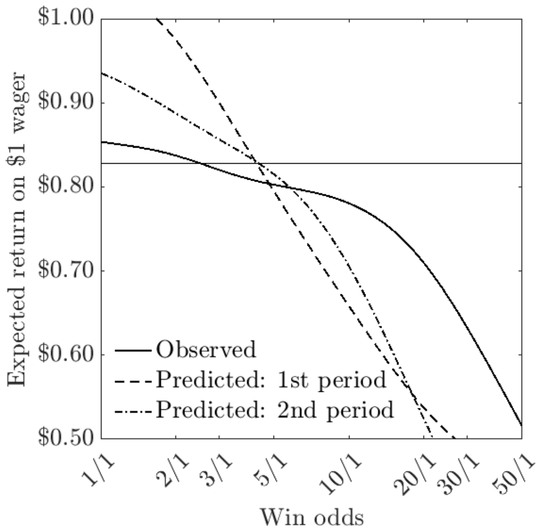
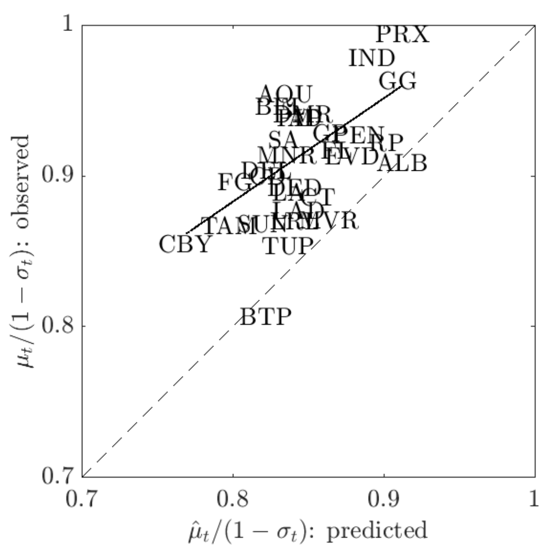


Figure A5: Replications of Figures 9a (a) and 10 (b) when using race variables to estimate the empirical win rates, p.

(a) Odds and expected returns



(b) Mean returns



B Estimation of empirical win rates

Consider a multinomial logistic regression in which the empirical win rate of horse i in race r is modeled as:

$$p_{i,r} \equiv P(i \text{ wins } r | q_{i,r}) = \frac{\exp(\beta_0 + \beta_1 q_{i,r})}{\sum_{j \in \mathcal{R}} \exp(\beta_0 + \beta_1 q_{j,r})} \tag{9}$$

The parameters β_0 and β_1 are found by maximizing the log-likelihood:

$$\log \mathcal{L} = \sum_{r=1}^{R} \sum_{i \in \mathcal{R}} \log p_{i,r} \cdot \mathbb{1}\{i \text{ won } r\},$$

where R is the number of races.

In a two-layer, fully-connected, feed-forward neural network, the data in (9) are replaced with a K-length vector of weights, ω. These weights are then modeled using the same logistic formulation:[49]

$$p_{i,r} = \frac{\exp(\beta_0^{(1)} + \beta^{(1)} \cdot \omega)}{\sum_{j \in \mathcal{R}} \exp(\beta_0^{(1)} + \beta^{(1)} \cdot \omega)} \qquad \omega_k = \frac{\exp(\beta_0^{(0)} + \beta_1^{(0)} q_{i,r})}{\sum_{j \in \mathcal{R}} \exp(\beta_0^{(0)} + \beta_1^{(0)} q_{j,r})},$$

where the superscript on the β parameters denotes the layer of the network, with 0 indexing the initial layer, where the data enter. The network can be generalized to an arbitrary number of layers in which the output of the initial layer is $\omega^{(1)}$, the output of the next layer is $\omega^{(2)}$, and so on. As with logistic regression, the parameters are trained by maximizing the log-likelihood using (some variant of) gradient descent. However, instead of passing over the full training data before updating the parameters, as is typically done with logistic regression, neural nets are trained using (some variant of) *stochastic* gradient descent, in which examples from the training data are sampled in random order, and updates are made every minibatch, or subset of examples.

Increasing K and/or the number of layers allows the neural net to conform to arbitrary functions. The downside of this flexibility is that even neural nets with small K and few layers can overfit—i.e., make predictions that perform poorly in a holdout set. Overfitting can be attenuated by stopping training early, or by choosing hyperparameters (e.g., K or the number of layers) to maximize the objective function on a holdout set. Srivastava et al.

[49] Logistic regression uses the sigmoid as the link function (termed the activation function in machine learning). However, different link functions, such as the inverse tangent are often used. We use the sigmoid.

(2014) show that overfitting can be better addressed through a regularization technique called dropout. In training, a fair coin is flipped for every $\omega_k^{(\ell)}$, once per minibatch. If the coin comes up heads, $\omega_k^{(\ell)}$ is set to 0; otherwise, the weight is left unchanged. With dropout, large networks perform well on holdout data without early stopping or hyperparameter tuning.

We estimate the largest network that we can train in a reasonable amount of time: 4 layers, $K = 64$, and regularized with dropout in the manner described above. We train each network for 2000 complete passes over the training data, using the ADAM stochastic gradient descent algorithm (Kingma and Ba, 2014) with a learning rate of 0.01 and a minibatch size of 96. In addition to the implied probability q, we include $\log q$, as well as two race-specific variables derived from $\{q\}_{i\in\mathcal{R}}$: the number of horses in the race and the inverse of the normalization constant in (1), which is one minus the takeout rate implied by the morning-line odds. Instead of letting the race-specific variables enter directly, we use the two eigenvectors, which are uncorrelated. This improves model fit, both in the training data and the holdout race.

C Proofs

C.1 Proof of Lemma 1

The second-period equilibrium conditions on the expected returns to arbitrageurs imply an expression for $\sum_{i=1}^{N} x_i$, the total amount wagered by arbitragers. Let $Q_+ \equiv \sum_{i=1}^{N} q_i \mathbb{1}\{x_i > 0\}$ and $P_+ = \sum_{i=1}^{N} p_i \mathbb{1}\{x_i > 0\}$, the subjective probability—held by noise traders and arbitrageurs, respectively—of an outcome that attracts positive bets by arbitrageurs. Then,

$$\sum_{i=1}^{N} x_i = \frac{(1-\sigma)P_+ - (1-r)Q_+}{(1-r) - (1-\sigma)P_+}$$

Substituting this expression into the track's profit function and solving the first-order condition yields the profit-maximizing rebate, r^*:

$$r^* = 1 - (1-\sigma)\left[P_+ + \sqrt{\frac{P_+(1-P_+)(1-Q_+)}{Q_+}}\right]$$

Substituting $\sum x$ and r^* into the first equilibrium condition and solving for x_i yields:

$$x_i = p_i\sqrt{\frac{Q_+(1-Q_+)}{P_+(1-P_+)}} - q_i, \;\; \forall i \text{ s.t. } x_i > 0$$

Hence, $x_i > 0$ if and only if:

$$\left(\frac{p_i}{q_i}\right)^2 > \frac{P_+(1-P_+)}{Q_+(1-Q_+)}$$

Without loss of generality, reorder the outcome indices i such that:

$$\frac{p_1}{q_1} \geq \frac{p_2}{q_2} \geq \cdots \geq \frac{p_i}{q_i} \geq \frac{p_{i+1}}{q_{i+1}} \geq \cdots \geq \frac{p_N}{q_N}$$

As a result, $Q_+ \equiv Q_m = \sum_{i=1}^m q_i$, where $x_i > 0$ for $i \leq m$ and $x_i = 0$ for $m < i \leq N$. Similarly, $P_+ \equiv P_m = \sum_{i=1}^m p_i$.

An equilibrium exists if there exists an index m such that $x_i > 0$ for $i \in \{1, \ldots, m\}$, and $x_i = 0$ for $i \in \{m+1, \ldots, N\}$. Given that p_i/q_i is decreasing in i by construction, this equilibrium condition can be written as:

$$\exists m \in \{1, \ldots, N-1\} \;\text{ s.t. } \left(\frac{p_m}{q_m}\right)^2 > \frac{P_m(1-P_m)}{Q_m(1-Q_m)} \geq \left(\frac{p_{m+1}}{q_{m+1}}\right)^2 \tag{10}$$

C.2 Proof of Corollary 1

We prove that $P_i \geq Q_i$ for all $i \in \{1, \ldots, N\}$. Since $P_N = Q_N = 1$, the result obtains if $P_i \geq Q_i$ is weakly decreasing in i—i.e., if:

$$\frac{P_i}{Q_i} \geq \frac{P_{i+1}}{Q_{i+1}} = \frac{P_i + p_{i+1}}{Q_i + q_{i+1}},$$

or equivalently, if:

$$\frac{P_i}{Q_i} \geq \frac{p_{i+1}}{q_{i+1}}$$

Since $p_i/q_i \geq p_{i+1}/q_{i+1}$, it is sufficient to show that:

$$\frac{P_i}{Q_i} \geq \frac{p_i}{q_i}$$

We prove this by induction. Observe that $P_1/Q_1 = p_1/q_1$ by definition. Assume that $P_i/Q_i \geq p_i/q_i$. It remains to be shown that:

$$\frac{P_{i+1}}{Q_{i+1}} = \frac{P_i + p_{i+1}}{Q_i + q_{i+1}} \geq \frac{p_{i+1}}{q_{i+1}},$$

or equivalently, that:

$$\frac{P_i}{Q_i} \geq \frac{p_{i+1}}{q_{i+1}},$$

which follows from the induction assumption and the ordering of the indices. □

C.3 Proof of Proposition 1

We prove that if $p_1 > q_1$, then the equilbrium condition in (10) holds. Namely, there exists an index $m \in \{1, \ldots, N-1\}$ such that $(p_m/q_m)^2 > \lambda_m \geq (p_{m+1}/q_{m+1})^2$, where:

$$\lambda_i \equiv \frac{P_i(1-P_i)}{Q_i(1-Q_i)}$$

Observe that $(p_i/q_i)^2$ is weakly decreasing in i, which follows from the fact that p_i/q_i is weakly decreasing in i by construction. The following conditions on λ_i complete the proof:

1. λ_i is weakly decreasing in i.

2. $\lambda_i < (p_1/q_1)^2$.

3. $\lambda_i > (p_N/q_N)^2$.

The logic is that $(p_i/q_i)^2$ is collapsing on λ_i, which creates a crossing point. We offer a proof by contradiction. Assume that there is no index $i \in \{1, \ldots, N-1\}$ such that $(p_i/q_i)^2 > \lambda_i \geq (p_{i+1}/q_{i+1})^2$. Given condition 3, it then must be that $\lambda_{N-1} \geq (p_{N-1}/q_{N-1})^2$; otherwise, $m = N-1$ would satisfy (10). And given condition 1, this inequality must hold generally—i.e., $\lambda_i \geq (p_i/q_i)^2$; otherwise, $m = i$ would satisfy (10). But this violates condition 2.

If condition 1 holds, then condition 2 follows from the assumption that $p_1 > q_1$, and condition 3 follows from $p_N < q_N$, which is implied by $p_1 > q_1$. Specifically, $p_1 > q_1$ implies:

$$\lambda_1 = \frac{p_1(1-p_1)}{q_1(1-q_1)} < \left(\frac{p_1}{q_1}\right)^2$$

And $p_N < q_N$ implies:

$$\lambda_{N-1} = \frac{(1-p_N)p_N}{(1-q_N)q_N} > \left(\frac{p_N}{q_N}\right)^2$$

We now prove condition 1. λ_i is weakly decreasing in i if $\lambda_i \geq \lambda_{i+1}$, or if:

$$\frac{P_i(1-P_i)}{Q_i(1-Q_i)} \geq \frac{P_{i+1}(1-P_{i+1})}{Q_{i+1}(1-Q_{i+1})},$$

Observe that:

$$\begin{aligned}\frac{P_{i+1}(1-P_{i+1})}{Q_{i+1}(1-Q_{i+1})} &= \frac{(P_i+p_{i+1})(1-P_i-p_{i+1})}{(Q_i+q_{i+1})(1-Q_i-q_{i+1})}\\ &= \frac{P_i(1-P_i)+p_{i+1}(1-P_i-P_{i+1})}{Q_i(1-Q_i)+q_{i+1}(1-Q_i-Q_{i+1})}\end{aligned}$$

Hence, λ_i is weakly decreasing in i if:

$$\frac{P_i(1-P_i)}{Q_i(1-Q_i)} \geq \frac{p_{i+1}(1-P_i-P_{i+1})}{q_{i+1}(1-Q_i-Q_{i+1})},$$

or equivalently, if:

$$P_i\left(1-\frac{Q_{i+1}}{1-Q_i}\right)q_{i+1} \geq Q_i\left(1-\frac{P_{i+1}}{1-P_i}\right)p_{i+1} \tag{11}$$

If $Q_{i+1} < 1-Q_i$, then (11) can be rewritten as:

$$\frac{P_i}{Q_i} \geq \frac{p_{i+1}}{q_{i+1}}\left(1-\frac{P_{i+1}}{1-P_i}\right)\Big/\left(1-\frac{Q_{i+1}}{1-Q_i}\right)$$

Since $P_i/Q_i \geq p_{i+1}/q_{i+1}$ (see Section C.2), the above inequality holds if:

$$1-\frac{P_{i+1}}{1-P_i} \leq 1-\frac{Q_{i+1}}{1-Q_i},$$

or if $P_{i+1}/Q_{i+1} \geq (1-P_i)/(1-Q_i)$. From Corollary 1, $P_{i+1}/Q_{i+1} \geq 1 \geq (1-P_i)/(1-Q_i)$. Thus, the inequality holds.

If $Q_{i+1} > 1 - Q_i$, then (11) can be rewritten as:

$$\frac{1-P_i}{1-Q_i} \leq \frac{p_{i+1}}{q_{i+1}}\left(1 - \frac{1-P_{i+1}}{P_i}\right)\Big/\left(1 - \frac{1-Q_{i+1}}{Q_i}\right)$$

Since $(1-P_i)/(1-Q_i) \leq p_{i+1}/q_{i+1}$ (see Section C.2), the above inequality holds if:

$$1 - \frac{1-P_{i+1}}{P_i} \geq 1 - \frac{1-Q_{i+1}}{Q_i},$$

or if $P_i/Q_i \geq (1-P_{i+1})/(1-Q_{i+1})$. From Corollary 1, $P_i/Q_i \geq 1 \geq (1-P_{i+1})/(1-Q_{i+1})$. Thus, the inequality holds.

Finally, if $Q_{i+1} = 1 - Q_i$, then the inequality in (11) holds if $P_{i+1} \geq 1 - P_i$. From Corollary 1, $P_{i+1} \geq Q_{i+1}$ and $1 - Q_i \geq 1 - P_i$. Thus, the inequality holds. □

C.4 Proof of Lemma 2

The first-order condition for the broker's profit, given $\alpha = 1/2$, is:

$$\frac{\partial \pi(q|p) - C_{\frac{1}{2}}(q|p)}{\partial q} = 0 \iff \left(\sqrt{pq} + \sqrt{(1-p)(1-q)}\right)^2 = \frac{c}{1-\sigma},$$

where $0 < c < 1 - \sigma$ is the coefficient of proportionality in the cost function. Substituting the above expression into the profit function yields the maximal profit:

$$\pi(q|p) - C_{\frac{1}{2}}(q|p) = \sigma + (1-\sigma)\left(1 - \frac{c}{1-\sigma}\right) + 2c\log\left(\sqrt{\frac{c}{1-\sigma}}\right),$$

which does not depend on q. □

C.5 Proof of Proposition 2

Consider a prediction $q = p + \epsilon$ for $\epsilon \in (0, 1-p]$. Let $\epsilon^*(\alpha)$ be such that $C_\alpha(p+\epsilon|p) = C_\alpha(p - \epsilon^*(\alpha)|p)$ for some ϵ. We show that as α increases, so too does $p - \epsilon^*(\alpha)$. Hence, an increase in α increases the cost of underestimation more slowly than the cost of overestimation. Since under- and overestimation are equally profitable when $\alpha = \frac{1}{2}$, it must be the case that underestimation is more profitable when $\alpha > \frac{1}{2}$.

We begin by deriving $\epsilon^*(\alpha)$ when $\alpha = 1/2$, or ϵ^* for short, by solving $C_{\frac{1}{2}}(p + \epsilon|p) =$

$C_{\frac{1}{2}}(p-\epsilon^*|p)$:

$$\epsilon^* = \epsilon + 4(2p-1)\bigg(p(1-p) - \sqrt{p(1-p)(p+\epsilon)(1-p-\epsilon)} - \frac{\epsilon}{2}(2p-1)\bigg) \tag{12}$$

Given that $\epsilon > 0$, it follows that $\epsilon^* > \epsilon$.

Now consider predictions that generate equivalent costs for a generic α:

$$\begin{aligned}\frac{c}{\alpha-1}\log\left(\frac{p^\alpha}{(p+\epsilon)^{a-1}} + \frac{(1-p)^\alpha}{(1-p-\epsilon)^{a-1}}\right) \\ = \frac{c}{\alpha-1}\log\left(\frac{p^\alpha}{\big(p-\epsilon^*(\alpha)\big)^{a-1}} + \frac{(1-p)^\alpha}{\big(1-p+\epsilon^*(\alpha)\big)^{a-1}}\right),\end{aligned}$$

or equivalently:

$$\begin{aligned}\log\left(\frac{p}{1-p}\right) = \frac{1}{\alpha}\bigg(&\log\left[\big(1-p+\epsilon^*(\alpha)\big)^{1-\alpha} - (1-p-\epsilon)^{1-\alpha}\right] \\ &- \log\left[(p+\epsilon)^{1-\alpha} - \big(p-\epsilon^*(\alpha)\big)^{1-\alpha}\right]\bigg)\end{aligned} \tag{13}$$

Define the right hand side as $G\big(\alpha, \epsilon^*(\alpha)\big)$. By the Implicit Function Theorem,

$$\frac{\partial\epsilon^*(\alpha)}{\partial\alpha} = -\frac{\partial G\big(\alpha,\epsilon^*(\alpha)\big)}{\partial\alpha}\bigg/\frac{\partial G\big(\alpha,\epsilon^*(\alpha)\big)}{\partial\epsilon^*(\alpha)}$$

We evaluate this derivative at $\alpha = 1/2$ and ϵ^*. Begin with the denominator:

$$\frac{\partial G\big(\alpha,\epsilon^*(\alpha)\big)}{\partial\epsilon^*(\alpha)} = \frac{1}{\sqrt{1-p+\epsilon^*}\big(\sqrt{1-p+\epsilon^*} - \sqrt{1-p-\epsilon}\big)} - \frac{1}{\sqrt{p-\epsilon^*}\big(\sqrt{p+\epsilon} - \sqrt{p-\epsilon^*}\big)}$$

It follows from $\epsilon^* > 0$ that $\partial G\big(\alpha,\epsilon^*(\alpha)\big)/\partial\epsilon^*(\alpha) < 0$. Hence, $\epsilon^*(\alpha)$ is increasing in α if the numerator is positive. The numerator is:

$$\begin{aligned}\frac{\partial G\big(\alpha,\epsilon^*(\alpha)\big)}{\partial\alpha} = 2\Bigg[&\frac{\sqrt{p+\epsilon}\log(p+\epsilon) - \sqrt{p-\epsilon^*}\log(p-\epsilon^*)}{\sqrt{p+\epsilon} - \sqrt{p-\epsilon^*}} \\ &- \frac{\sqrt{1-p+\epsilon^*}\log(1-p+\epsilon^*) - \sqrt{1-p-\epsilon}\log(1-p-\epsilon)}{\sqrt{1-p+\epsilon^*} - \sqrt{1-p-\epsilon}} \\ &+ \log(p+\epsilon) + \log(1-p-\epsilon) - \log(p-\epsilon^*) - \log(1-p+\epsilon^*)\Bigg]\end{aligned}$$

Since the natural log is a concave function, and since $\epsilon^* > \epsilon$, it follows that $\log(p+\epsilon) + \log(1-p-\epsilon) \geq \log(p-\epsilon^*) + \log(1-p+\epsilon^*)$. Hence, $\partial G\big(\alpha, \epsilon^*(\alpha)\big)/\partial\alpha > 0$ if:

$$\frac{\sqrt{1-p+\epsilon^*} - \sqrt{1-p-\epsilon}}{\sqrt{p+\epsilon} - \sqrt{p-\epsilon^*}} > \frac{\sqrt{1-p+\epsilon^*}\log(1-p+\epsilon^*) - \sqrt{1-p-\epsilon}\log(1-p-\epsilon)}{\sqrt{p+\epsilon}\log(p+\epsilon) - \sqrt{p-\epsilon^*}\log(p-\epsilon^*)}$$

The left-hand side equals $\sqrt{p/(1-p)}$, from (13). Since $\sqrt{p/(1-p)} \geq 1$, it follows that $\partial G\big(\alpha, \epsilon^*(\alpha)\big)/\partial\alpha > 0$ if:

$$\sqrt{p+\epsilon}\log(p+\epsilon) + \sqrt{1-p-\epsilon}\log(1-p-\epsilon) > \\ \sqrt{p-\epsilon^*}\log(p-\epsilon^*) + \sqrt{1-p+\epsilon^*}\log(1-p+\epsilon^*),$$

or if $y(\epsilon) > y(-\epsilon^*)$, where $y(x) = \log(p+x)\sqrt{p+x} + \log(1-p-x)\sqrt{1-p-x}$ for $x \in (0, 1-p]$. Observe that y is continuous and has the following properties:

1. $y''(x) > 0$
2. $y'(x) = 0$ when $x = 1/2 - p$
3. $y(x) = y(1-2p-x)$

The first two properties imply that $y'(x) > 0$ when $x > 1/2 - p$, and $y'(x) < 0$ when $x < 1/2 - p$. The third property implies that $y(\epsilon) > y(x)$ if $\epsilon > x > 1 - 2p - \epsilon$. Hence, $y(\epsilon) > y(-\epsilon^*)$ if $\epsilon > -\epsilon^* > 1 - 2p - \epsilon$. The first inequality is true because $\epsilon^* > 0$, and the second inequality follows from (12). $\square$

D Beliefs in exotic pools

We first consider place and show wagers, which pay out if the chosen horse finishes in the top 2 or 3 positions, respectively. The probability of a place wager on horse i paying out is:

$$p_{i,\text{place}} = p_{i,\text{win}} + \sum_{j \neq i} p_{j,\text{win}} \frac{p_{i,\text{win}}}{1 - p_{j,\text{win}}}$$

where the second term is the probability of horse i finishing second. Beliefs for noise traders, $q_{i,\text{place}}$, can be calculated in a corresponding manner. Similarly, the probability of a show wager on horse i paying out is:

$$p_{i,\text{show}} = p_{i,\text{place}} + \sum_{j \neq i} \sum_{k \neq (i,j)} p_{j,\text{win}} \frac{p_{k,\text{win}}}{1 - p_{j,\text{win}}} \frac{p_{i,\text{win}}}{1 - p_{j,\text{win}} - p_{k,\text{win}}}$$

where the second term is the probability of horse i finishing third.

We also consider a range of exotic wagers. Bettors may wager on the order of the first n horses in a single race, as in exacta and trifecta pools, which we denote by order-n. The probability of a sequence, $\vec{v}$, is:

$$p_{\text{order-}n}(\vec{v}) = p_{\vec{v}_1,\text{win}} \times \frac{p_{\vec{v}_2,\text{win}}}{1 - p_{\vec{v}_1,\text{win}}} \times \cdots \times \frac{p_{\vec{v}_n,\text{win}}}{1 - p_{\vec{v}_1,\text{win}} - \cdots - p_{\vec{v}_{n-1},\text{win}}}$$

A variant of the exacta (i.e., order-2) is the quinella, in which the bettor predicts the first two horses regardless of order. Assuming conditional independence, the probability of a quinella bet on (i, j) paying out is:

$$p_{\text{quin}}(i, j) = p_{i,\text{win}} \frac{p_{j,\text{win}}}{1 - p_{i,\text{win}}} + p_{j,\text{win}} \frac{p_{i,\text{win}}}{1 - p_{j,\text{win}}}$$

This is equivalent to the probability that an exacta bet on either (i, j) or (j, i) pays out.

Bettors may also wager on the winner of n consecutive races, as in daily-double and pick-3 pools, which we denote by pick-n. Assuming independence between races, the probability of a sequence, $\vec{v}$, is:

$$p_{\text{pick-}n}(\vec{v}) = \prod_{i=1}^{n} p^{(i)}_{\vec{v}_i,\text{win}}$$

where the superscript indexes the race.