Asymmetric Risk Information and Derivative Markets

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Abstract

In this paper, we develop a model in which investors possess private information on both the expected level of a stock's payoffs and their risk. These investors may trade in both the stock and a derivative whose payoff is a function of the riskiness of the stock. The model suggests that risk uncertainty affects how investors trade on their mean information, and ties the equity risk premium to the derivative price. Unlike prior rational expectation models with derivatives, the derivative price serves a valuable informational role because of the fact that investors possess risk information.

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1 Introduction

There is widespread interest amongst academics, practitioners, and regulators regarding the drivers of trading volume and stock returns when investors have diverse private information. Since Grossman and Stiglitz (1980), rational expectations models with noise traders have served as a foundation for understanding how markets aggregate investors' private information. These models take a somewhat narrow view on the type of information possessed by investors, assuming that this information informs them only regarding a security's expected future performance.[1] However, the abundance of trade in derivative instruments whose values are directly impacted by the volatility of their underlyings' payoffs, such as options and variance swaps, suggests that the information possessed by traders concerns not only average future payoffs, but also their risk (e.g., Ni, Pan, and Poteshman (2008)). Moreover, many of the means through which traders acquire private information may lead them to learn about a stock's riskiness in addition to its expected performance. For instance, an investor who is privately aware that a firm has an impending announcement may know that the firm's stock price will move, but still face uncertainty regarding the direction of this movement. Similarly, a party who privately knows the amount of capital available to a firm has a more accurate assessment of the firm's future risky investment than an uninformed investor. This raises the question of how uncertainty over a stock's risk and asymmetric information regarding this risk affect prices and trading volume in the equity and derivative markets.

In order to address this question, we build a rational expectations model in which traders are risk averse and face uncertainty regarding both the mean and variance of a stock's payoffs. These traders possess diverse private information on each of these components, i.e., they each possess both "mean" and "risk" information, and trade in equity and a derivative security whose payoff is exclusively a function of the riskiness of the stock. In particular, the

[1] While some prior literature has examined rational expectations models with non-normal distributions, and hence, signals that lead to updating on moments other than the first, even in these frameworks, signals order the posterior distributions in the sense of first-order stochastic dominance (e.g., Breon-Drish (2015a), Vanden (2006)).

mean information received by investors informs them regarding the first moment, or location parameter, of the stock's payoffs, while their risk information informs them regarding the second moment, or dispersion parameter, of the stock's payoffs. The rational expectations literature provides the critical insight that prices aggregate diverse private information, and hence, offer traders valuable information. In line with this literature, in addition to taking into account their private information when determining their portfolios, investors in our model also take into account the information contained in the equilibrium stock and derivative prices, which are each affected by noise trade. In practice, it appears that both equity and derivative prices provide useful information to investors. In particular, implied volatility is frequently used as a measure of the market's assessment of the return variance, suggesting that option prices aggregate investors' information regarding future volatility. This is difficult to reconcile with existing theoretical models of derivative trade (Brennan and Cao (1998), Vanden (2008)), which find that derivative prices serve no informational role, but model the stock's risk as common knowledge. In our model, both the derivative and equity prices provide investors with valuable information.

Introducing risk uncertainty and private risk information in a setting with both an equity and derivative leads to two foundational results. First, the presence of uncertainty over the stock's risk affects how investors trade on their mean information. Prior models of trade with known risk demonstrate that investors trade on their beliefs about a stock's expected payoffs in equity, but not in derivatives (Brennan and Cao (1998), Cao and Ou-Yang (2008)). On the other hand, in the face of risk uncertainty, derivatives serve as a form of insurance against fluctuations in the riskiness of the stock's payoffs. When the riskiness of the stock's payoffs is high, a risk-averse investor who holds an equity position has heightened marginal utility.[2] Therefore, they have a desire to "hedge" against risk uncertainty by purchasing a security that pays off in this state; the derivative security fills precisely this role. As a result, investors with optimistic mean information not only purchase the equity, but also

[2] More specifically, an investor with prudent preferences, i.e., those characterized by a utility function with a positive third derivative, will exhibit this behavior.

purchase the derivative. Second, the price of the derivative security, which is affected by both private information and noise trade, directly impacts the risk premium in the equity market. Intuitively, the price of the derivative reflects the cost to hedging the risk uncertainty induced by a position in the equity. Thus, the price of the equity is directly linked to the price of the derivative. The main goal of the paper is to analyze these two forces within the context of a rational expectations equilibrium.

In the model, there is a unique equilibrium in which the derivative price signals *only* risk information to investors and the equity price signals *only* mean information. This equilibrium possesses several noteworthy features. Again, investors trade on their *mean* information in both the stock and the derivative, since the derivative insures against the risk uncertainty created by speculative positions in the stock. Another way to view this phenomenon is that by purchasing the derivative, an investor holding equity is wealthier when they face greater levels of risk, i.e., the distribution of their payoffs exhibits positive skewness, which they favor. A second feature of the equilibrium is that investors take into account their *risk* information when determining their position in the derivative, but not when choosing their position in the equity. Intuitively, when the investor trades on risk information in the derivative, the only risk they face is that their information is inaccurate. On the other hand, if the investor were to trade on risk information in the stock, they would face both the risk that their information is inaccurate, and the risk that price moves against them. In sum, there are two components to the investor's demand in the derivative market: a speculative risk-information component, and a risk-uncertainty hedging component.

Despite the fact that investors trade on both mean and risk information in the derivative market, the equilibrium derivative price provides investors with risk information only. Investors trade on their mean information in the derivative due to their desire to hedge risk uncertainty, which is an increasing function of their equity demand. While any given individuals' equity demand increases in their mean information, the derivative price is a function of the *aggregate* equity demand of investors, which is fixed by the market clearing condition. Moreover, despite the fact that investors' equity demands are unaffected by their individual risk information, the equity price is a function of their *average* risk information. Intuitively, investors' equity demands decrease in the derivative price, as a higher derivative price makes it costlier to hedge risk uncertainty. Consequently, the equity price decreases in the derivative price, which is a function of the average risk information received by investors. The unique equilibrium is the solution to a fixed point problem: investors' equity demands are a function of the derivative price, and the derivative price itself is affected by investors' equity demands, which determine their desire to hedge risk uncertainty.

After characterizing the unique equilibrium, we consider four applications of the model. To begin, we demonstrate how the two foundational results discussed above lead to relationships between price changes and trading volume both within and across the two markets. The well-known positive relationship between trading volume in stocks and contemporaneous stock returns arises in the model (Karpoff (1987)). Furthermore, trading volume in the derivative is positively associated with contemporaneous stock returns and trading volumes in both the stock and derivative are negatively associated with contemporaneous derivative returns. To understand these results, suppose that the derivative price rises, which occurs when investors receive information that suggests the stock is risky or when noise traders purchase the derivative. By increasing the cost to hedging risk uncertainty, this leads the equity price to fall and reduces the intensity with which investors trade on their mean information. Hence, the equity price and equity trading volume are positively associated, and the derivative price and equity trading volume are negatively associated. Furthermore, as investors trade less intensely on their mean information, their equity positions converge, leading them to have similar desires to hold the derivative for hedging purposes. This causes the derivative trading volume to decline, such that trading volume in the derivative market is negatively associated with the derivative price and positively associated with the equity price.

Next, we study how dispersion in investors' beliefs over the mean of the equity's payoffs impacts the equity and derivative prices. Prior noisy rational expectations literature suggests that holding fixed the quality of investors' information and their average belief about an asset's expected payoffs, belief dispersion has no impact on the asset's price (Banerjee (2011), Lambert, Leuz, and Verrecchia (2012)).[3] We find that this result no longer holds in the presence of risk uncertainty. Intuitively, investors are averse to uncertainty over the riskiness of the stock's payoffs, which manifests as kurtosis in the distribution of the stock's payoffs. This causes an investor's distaste for risk to grow more rapidly with the size of their demand for the stock. Dispersion in investors' equilibrium beliefs leads optimistic investors to hold large equity positions, and pessimistic investors to hold small equity positions. The increased risk premium charged by optimistic investors exceeds the decrease in the risk premium charged by pessimistic investors, decreasing the stock price. Moreover, belief dispersion increases derivative prices. Investors' demand to hedge risk uncertainty is convex in their equity demands. Consequently, the increase in hedging demands of optimistic investors outweighs the decrease in the hedging demands of pessimistic investors, causing the aggregate demand curve for the derivative to shift right.

The fact that belief dispersion affects stock and derivative prices also has implications for the relationship between the quality of investors' private mean information and stock returns. Prior literature has suggested that when investors' private information grows more precise, they face less uncertainty and risk premia shrink. In our model, modifying the quality of investors' private mean information has two effects on the risk premium, which can work against one another. First, the standard effect is manifest: more precise private information reduces the risk premium that investors charge. Second, higher quality private information can cause investors to place more weight on their private signals, increasing the

[3]The analysis in Banerjee (2011) states that belief dispersion will increase expected returns in a noisy rational expectations setting. Note, however, that this is only the case when belief dispersion is created through a change in the precision of investors' information (see Proposition 1 of his paper). That is, the analysis he considers is not a ceteris paribus modification of belief dispersion, but rather, a change in the underlying information structure that creates belief dispersion. In his model, a ceteris paribus modification of belief dispersion would have no impact on prices: he states, "investor disagreement does not affect prices (while the average beliefs do) (pg. 38)."

dispersion in their beliefs and hence the stock's risk premium. In some cases, higher quality private information reduces the stock price and increases the derivative price.

In the third application, we study how shocks to the derivative price impact the efficiency of the stock price with respect to investors' mean information. We find that shocks to the derivative price, such as noise trade in the derivative market or changes in the quality of investors' risk information, have spillover effects on the stock market. Again, when the derivative price is high, investors trade less intensely on their mean information. This leads the stock price to become more sensitive to noise trade, rather than information-related trade, and heightens investors' perception of the stock's riskiness. Moreover, the effect of the original shock to the derivative demand is amplified, as a less efficient stock price increases investors' belief dispersion, resulting in an even higher derivative price.

Finally, we relate the variance risk premium, i.e., the spread between investors' expectation of future variance and their pricing of the variance, to returns in the stock market, dispersion in investors' equilibrium beliefs, their private information quality, and trading volume in the stock and derivative markets. Similar to prior literature, we find that there exists a deterministic relation between the variance risk premium and returns in the equity market (Bollerslev, Tauchen, and Zhou (2009), Buraschi et al. (2014)). Moreover, dispersion in investors' equilibrium beliefs regarding expected future cash flows increases the variance risk premium. Consequently, this premium increases in prior uncertainty regarding expected future cash flows, and is non-monotonic in the quality of investors' mean information. Finally, we find that the variance risk premium is negatively correlated with trading volume in the stock and derivative markets.

Related Literature. The primary contribution of our paper is to study a noisy rational expectations model in which i) investors trade on risk information, and ii) investors trade on mean information given the presence of risk uncertainty and a derivative market. While allowing for investors to possess private risk information is itself new to the literature, the most interesting implications of the model come from considering trading volume in equities and derivatives and the formation of a derivative price.

Our model fits into the strand of options pricing literature that takes assets' cash flows, rather than returns, as the primitive, and simultaneously derives stock and options returns in equilibrium (e.g., Brennan and Cao (1998), Cao (1999), Cao and Ou-Yang (2008), Vanden (2008)). The classic Black-Scholes option pricing model and its extension to stochastic variance (Heston (1993)) take the distribution of returns as a primitive and derive option prices under a no-arbitrage condition. This approach is practical given an empirically observed return distribution, but, in the words of Vanden (2008), "does very little to enhance our understanding of how the economy's primitives, such as information quality, affect the options market." In particular, taking returns as exogenous is limiting as the distribution of returns is affected by the presence of an option when investors have private information over cash flow risk.[4]

Brennan and Cao (1998), Cao (1999), and Vanden (2008) also study noisy rational expectations models in which investors may trade in a stock or options written on the stock. In these models, options complete the market when investors have heterogenous information quality. Investors take *deterministic* positions in the option based on the precision of their information relative to the average precision of all investors, and hence, option prices provide no information to investors. Chabakauri, Yuan, and Zachariadis (2016) study a rational expectations model in which investors may trade in a full set of contigent claims, also finding that derivative securities are informationally irrelevant. Buraschi and Jiltsov (2006) build a continuous time model in which investors have differences of opinion over the distribution of new information concerning the drift in an asset's dividend process. These investors can trade in the stock and options; they show that differences of opinion can impact the option prices. Back (1993), Biais and Hillion (1994), and Easley, O'Hara, and Srinivas (1998) study strategic trade in stock and options. Oehmke and Zawadowski (2015) analyze a model in

[4] The option is not a redundant security as in Black and Scholes (1973) for two reasons: i) its price serves as a signal of investors' information over cash flow risk and ii) since risk is uncertain, it serves to help complete the market.

which investors trade in derivatives due to differential trading costs, and analyze the effect of the derivative's introduction on the price of the underlying. In all of these models, investors' information orders their posteriors in the sense of first-order stochastic dominance, suggesting that they possess directional, rather than risk information.

Other models offer non-information related reasoning for why investors trade derivatives. Detemple and Selden (1991) show that in an incomplete market setting, investors who agree to disagree over an asset's risk trade in derivatives. Our work extends this intuition to a rational expectations setting. In a similar vein, Cao and Ou-Yang (2008) develop a model in which investors agree to disagree about the mean and precision of a signal, and may trade in a stock or an option. They find that disagreements about the mean lead to stock but not option trade and disagreements over precision lead to trade in both markets, contrasting with our model. This difference may be explained by the fact that in their model, the variance of cash flows is known and information is symmetric. This eliminates updating from option prices and the hedging component of investors' option demands, such that derivatives serve a different purpose. Leland (1980) shows that an investor's derivative demand is a function of how their risk aversion shifts with their wealth. Franke, Stapleton, and Subrahmanyam (1998) studies a setting in which investors face unhedgable background risks and finds that investors facing high background risk purchase derivatives from investors facing low background risk.

Derivative trade is often thought of as stemming from transaction costs and trading restriction-based motives (e.g., Anthony (1988), Diamond and Verrecchia (1987)). These motives are absent from our model, in which the derivative's value is a function only of the magnitude, and not the direction, of the underlying's return. Ni, Pan, and Poteshman (2008) develop an empirical measure of volatility-based trading in options markets and control for options trade relating to future changes in stock prices, and also examine option trade related specifically to straddles; this roughly captures derivative trade as we model it.

From a technical perspective, we borrow many of the techniques from Breon-Drish

(2015a,b).[5] Modeling the payoff to a derivative requires a distribution that is bounded below by zero. This requires a deviation from conventional noisy rational expectations models, which assume that payoffs are normally distributed and hence have no lower bound. The techniques of Breon-Drish (2015a,b) enable us to prove the existence and uniqueness and find the structure of an equilibrium in the derivative market.

2 Model

2.1 Assumptions

The model that we analyze is a one-period model of trade. We assume that the economy is populated by a unit continuum of informed investors indexed on $[0,1]$ with CARA utility $u(W) = -e^{-\frac{W}{\tau}}$. There are three securities in the economy. The first security is a risk free asset with payoff normalized to one, which is in unlimited supply. The second is a risky asset (the stock, or equity) that pays off a one time dividend of $\tilde{x}$ at the end of the period, with per capita supply of 0; a non-zero supply is absorbed into noise trade. We refer to the i^{th} trader's position in the stock as D_{Si}. In order for investors to possess information on both the stock's expected payoffs and its riskiness, both the mean and variance of its payoffs must be random. To accomplish this, we assume that $\tilde{x}$ has two components, one a location parameter with known variance, and a stochastic exposure to a random factor $\tilde{f}$. In particular, we assume that:

$$
\begin{aligned}
\tilde{x} &= \tilde{\mu} + \tilde{V}^{\frac{1}{2}} \tilde{f} \\
\text{where } \tilde{\mu} &\sim N\left(m, \sigma_\mu^2\right) \\
\text{and } \tilde{f} &\sim N(0,1)
\end{aligned}
$$

[5] Note the framework of Breon-Drish (2015) itself does not encompass the present model, in the sense that it does not allow for signals that concern only the riskiness of future cash flows. The reason is that the signal in his model orders the distribution in the sense of the monotone likelihood ratio property, and *a fortiori*, first order stochastic dominance.

where $\tilde{\mu}, \tilde{V}$, and $\tilde{f}$ are mutually independent random variables. This implies that fixing $\tilde{\mu}$ and $\tilde{V}$, $\tilde{x}$ is normally distributed with mean $\tilde{\mu}$ and variance $\tilde{V}$, that is, $\tilde{x}|\tilde{\mu}, \tilde{V} \sim N\left(\tilde{\mu}, \tilde{V}\right)$. We allow $\tilde{V}$ to have any distribution with a non-negative support Υ. By setting up the model in this fashion, we can clearly label private information regarding $\tilde{\mu}$ as information regarding expected payoffs, and private information regarding $\tilde{V}$ as risk information. All informed traders receive an information signal regarding $\tilde{\mu}$ and an information signal regarding $\tilde{V}$,[6] and these traders rationally use the stock and derivative prices as additional signals. In particular, the "mean" signal received by investor i is equal to $\tilde{\varphi}_i = \tilde{\mu} + \tilde{\varepsilon}_i$ and $\tilde{\varepsilon}_i \sim N(0, \sigma_\varepsilon^2)$. The "risk" signal received by investor i is equal to $\tilde{\eta}_i = \tilde{V} + \tilde{e}_i$ where $\tilde{e}_i \sim N(0, \sigma_e^2)$. The noise terms $\tilde{\varepsilon}_i$, $\tilde{e}_i$ are assumed independent of the other variables in the model.

The third and final security traded by investors has payoffs equal to the stochastic variance, $\tilde{V}$,[7] and has a per capita supply of zero. We refer to this security as the derivative and refer to the i^{th} trader's position in the derivative as D_{Di}. This approach to modeling the derivative deviates from prior literature, which studies derivatives with option-like payoffs, or payoffs that are a quadratic or logarithmic function of returns (Brennan and Cao (1998), Vanden (2008), Cao and Ou-Yang (2008)).[8] There are two ways to interpreting a security which pays off equal to the underlying variance of the stock's payoffs. First, it may be viewed as a mathematical approximation to an option. In particular, the payoff to a call or put option, or straddle, in expectation increases in the riskiness of the stock's payoffs, $\tilde{V}$, but, even for a fixed level of risk, this payoff still varies. Hence, by modeling the derivative's payoff as simply equal to $\tilde{V}$, we abstract from this second layer of uncertainty, but still capture the essential element that the expected payout to the derivative is greater when the variance of the stock's payoffs is larger. Second, in continuous time, the underlying stochastic variance in the process generating price would manifest deterministically as variance in

[6] The model is easily extended to the case in which some traders do not receive a variance signal. However, as we discuss later, all traders must have homogenous information precision regarding $\tilde{\mu}$ to ensure tractability.

[7] It is simple to accomodate the case in which $\tilde{y}$ also pays out the fixed component of the unconditional variance of $\tilde{x}$, σ_μ^2, but this adds unnecessary complexity to the expressions for price and demand.

[8] This approach is intractable in the case in which $\tilde{V}$ is stochastic.

price. Consequently, a variance swap, which pays off equal to the realized variance of prices, pays off exactly equal to the underlying stochastic variance. As a result, the derivative may heuristically be viewed as a variance swap.[9]

In order to close the model and prevent the stock price from fully revealing investors' mean information, we introduce noise traders in the two markets. First, we assume that there is a unit continuum of noise traders in the equity market whose aggregate demand is $\tilde{Z}_S \sim N\left(\bar{z}_S, \sigma_S^2\right)$ where $\bar{z}_S \leq 0$ and is independent of the other variables in the model. This ensures that the stock price reflects investors' private information only with noise. Next, we assume that these traders rationally take positions in the derivative market given their equity demand $\tilde{Z}_S$; we refer to these traders' demand in the derivative as D_Z. In the absence of this assumption, the derivative price provides informs the investors regarding $\tilde{Z}_S^2$, rendering the model intractable; the reason for this will become clear in the next section. It can be justified by viewing noise trade in the stock market as arising from liquidity traders who take a position in the equity and adjust their derivative demands accordingly. For simplicity of exposition, we treat the noise traders as symmetric to informed traders when trading in the derivative, by endowing them with risk signals $\tilde{\eta}_{Zi} = \tilde{V} + \tilde{e}_{Zi}$ where $\tilde{e}_{Zi} \sim N\left(0, \sigma_e^2\right)$ is independent of all other variables in the model.[10]

Finally, we assume that there are additional noise traders in the derivative market whose aggregate demand is $\tilde{Z}_D \sim N\left(0, \sigma_D^2\right)$, where $\tilde{Z}_D$ is independent of the other variables in the model. Given that noise traders in the stock market rationally take positions in the derivative, if there were not an additional source of noise in the derivative market, the

[9]We note that the model accomodates the case in which the derivative has both "delta" and "vega." First, note these options can be roughly approximated by taking a position in both the equity and derivative in my model. Second, suppose that the derivative payoff was instead linear in $\tilde{x}$ as well as $\tilde{V}$, i.e., its pay off was $\alpha\tilde{x} + \beta\tilde{V}$ for some $\alpha \in \Re$ and $\beta > 0$. In this case, the expression for the equity price would be materially unchanged as a result of the fact that the derivative is, on average, in zero net supply. However, trading volume in the asset would be a function of investors' risk information, as they would trade in the asset to neutralize the delta provided by a position in the derivative. The derivative price would equal $\alpha P_S + \beta P_D$ where P_S and P_D are the equity and derivative prices in my model, respectively.

[10]Relaxing this assumption would modify the derivative price, as there would be two types of investors' in the derivative who have information of varying precisions. However, allowing for heterogenous precisions in the derivative market would have no impact on the general tenure of the results, but would add complexity to the price expression (see Breon-Drish (2015b)).

derivative price would perfectly reveal investors' information regarding $\tilde{V}$.[11]

2.2 Equilibrium

We begin by characterizing a rational expectations equilibrium. Let P_D equal the equilibrium price of the derivative, and P_S the equilibrium price of the stock. Let $\Phi_i = \{\tilde{\varphi}_i, \tilde{\eta}_i, P_S, P_D\}$ represent investor i's information set. We start with the standard definition of a rational expectations equilibrium:

Definition 1 *A rational expectations equilibrium is a pair of functions P_S, P_D such that investors choose their demands to maximize their utility conditional on their information set:*

$$(D_{Si}(\Phi_i), D_{Di}(\Phi_i)) \in \arg\max_{d_{Si}, d_{Di}} E\left[-\exp\left(-\tau^{-1}\left(d_{Si}(\tilde{x} - P_S) + d_{Di}\left(\tilde{V} - P_D\right)\right)\right)|\Phi_i\right]$$

and, in all states, markets clear:

$$\begin{aligned} \int_0^1 D_{Si}(\Phi_i)\, di &= -\tilde{Z}_S \\ \int_0^1 D_{Di}(\Phi_i)\, di + D_Z &= -\tilde{Z}_D. \end{aligned}$$

Note that given the equilibrium definition, the investors' demands in the stock and derivative are allowed to depend on both the derivative price and the stock price. As a result, it is possible that the stock and derivative prices each contain information on both $\tilde{\mu}$ and $\tilde{V}$. We specialize slightly further in the equilibria we consider. In particular, we consider only equilibria in which the derivative price does not reveal any information incremental to the stock price regarding $\tilde{\mu}$ and the stock price does not reveal any information incremental to the derivative price regarding $\tilde{V}$. Technically, we take the following approach. Let $F_{P_S}(\cdot)$ represent the distribution function of P_S and $F_{P_D}(\cdot)$ represent the distribution function of

[11] Similar assumptions appear in Back (1993) and Easley, O'Hara, and Srinivas (1998).

P_D. We conjecture an equilibrium in which the derivative price is conditionally independent of $\tilde{\mu}$ given the stock price, i.e., $F_{P_D}(\cdot|P_S, \tilde{\mu}) = F_{P_D}(\cdot|P_S)$, and the stock price is conditionally independent of $\tilde{V}$ given the derivative price, i.e., $F_{P_S}\left(\cdot|P_D, \tilde{V}\right) = F_{P_S}(\cdot|P_D)$. This implies that investors use the stock price to update on expected payoffs and the derivative price to update on the riskiness of payoffs. We then show that given such a conjecture, the equilibrium stock price and derivative price indeed satisfy $F_{P_D}(\cdot|P_S, \tilde{\mu}) = F_{P_D}(\cdot|P_S)$ and $F_{P_S}\left(\cdot|P_D, \tilde{V}\right) = F_{P_S}(\cdot|P_D)$, demonstrating the existence of such an equilibrium. In fact, one needs only to conjecture that one of these two properties holds, and the other will follow in equilibrium. However, we have not been able to rule out the possibility of other equilibria.

We proceed in three steps: (i) we solve for equity demands and the equity price for a fixed derivative price and derivative demands; (ii) we solve for the derivative demands and derivative price for a fixed equity price and equity demands; (iii) we combine the two markets to show that a rational expectations equilibrium solves a fixed point problem that has a unique solution.

Beginning with the stock market, we follow the standard procedure of conjecturing a linear equilibrium:

$$P_S = \tilde{\alpha}_0 + \alpha_\mu \tilde{\mu} + \alpha_z \tilde{Z}_S \tag{1}$$

where, by the conjecture that $F_{P_S}\left(\cdot|P_D, \tilde{V}\right) = F_{P_S}(\cdot|P_D)$, $\tilde{\alpha}_0$ may depend upon $\tilde{V}$ and $\tilde{Z}_D$ only through P_D, and hence, is known to investors.[12] The following proposition summarizes the equilibrium equity demands and equity price, for a given derivative price P_D. In the appendix, we derive explicit expressions for α_0, α_μ, and α_z.

Proposition 1 *The investors' equity demands and the unique equilibrium equity price given*

[12] One may use the technique of Breon-Drish (2015b) to show that this conjectured linear equilibrium in the asset market is indeed unique amongst the class of equilibria in which the price function is continuous and satisfies a technical condition. We omit the formal argument here for sake of brevity.

a derivative price P_D satisfy:

$$D_{Si} = \tau \frac{E\left(\tilde{x}|\Phi_i\right) - P_S}{P_D + Var\left(\tilde{\mu}|\Phi_i\right)} \text{ and } \tag{2}$$
$$P_S = \int_0^1 E\left(\tilde{x}|\Phi_i\right) di + \tau^{-1}\tilde{Z}_S\left(P_D + Var\left(\tilde{\mu}|\Phi_i\right)\right).$$

To understand this expression, recall that in the classical mean-variance framework with known variance and no derivative security, investors' demands equal $\tau\frac{E(\tilde{x}|\Phi_i)-P_S}{Var(\tilde{x}|\Phi_i)}$. In the present setting, we again have a numerator equal to the expected payoff minus price. However, the denominator, which captures the investor's adjustment for risk, is now modified as there are two components of risk when trading in the stock: that of the uncertain mean, $\tilde{\mu}$, and that of the stochastic variance term, $\tilde{V}^{\frac{1}{2}}\tilde{f}$. As is the case in the classical framework, the variance of the uncertain mean component is added to the denominator since it follows a normal distribution. On the other hand, to account for the riskiness of the component of payoffs with an uncertain variance, $\tilde{V}^{\frac{1}{2}}\tilde{f}$, the denominator includes the price of the derivative security P_D, which in general is not simply equal to $Var\left(\tilde{V}^{\frac{1}{2}}\tilde{f}\right)$, since $\tilde{V}^{\frac{1}{2}}\tilde{f}$ is not normally distributed.

To provide an intuition for why investors discount the risk associated with $Var\left(\tilde{V}^{\frac{1}{2}}\tilde{f}\right)$ at the price of the derivative security, consider investor i's expected utility conditional on $\tilde{V}$ when their demands are (D_{Si}, D_{Di}):

$$E\left[-\exp\left(-\tau^{-1}\left(D_{Si}\left(\tilde{x} - P_S\right) + D_{Di}\left(\tilde{V} - P_D\right)\right)\right)|\Phi_i, \tilde{V}\right] \tag{3}$$
$$= -\exp\left[-\tau^{-1}D_{Si}\left(E\left(\tilde{\mu}|\Phi_i\right) - P_S\right) - \tau^{-1}D_{Di}\left(\tilde{V} - P_D\right) + \tau^{-1}D_{Si}^2\left(Var\left(\tilde{\mu}|\Phi_i\right) + \tilde{V}\right)\right] \tag{4}$$

Notice that the investor's expected utility is decreasing in a linear combination of the payoff to their derivative position, $D_{Di}\left(\tilde{V} - P_D\right)$, and the riskiness of the equity position, $D_{Si}^2\tilde{V}$. As a result, the exposure to $\tilde{V}$ created by a position in the stock, $\tau^{-1}D_{Si}^2\tilde{V}$ can effectively be hedged by taking a position in the derivative, which comes at the price of P_D. When P_D

rises, this exposure becomes costlier to hedge, and hence, investors will treat the stock as though it were riskier. The proof provided in the appendix demonstrates that this intuition continues to hold upon taking the expectation over $\tilde{V}$.

Note that the one-for-one relationship between the derivative and equity prices implies that even when driven by noise trade, increases in the derivative price lead to an increase in the equity risk premium. This effect only arises due to investors' preference for higher moments in their payoff distributions. In particular, since the investors' utility functions have a positive third derivative, they prefer skewed payoff distributions (Eeckhoudt and Schlesinger (2006)). By purchasing the derivative, an equity investor has a greater level of wealth when they face more risk, creating skewness in their payoff distribution.[13] If the investors had mean-variance preferences, rather than CARA utility, their demands for the stock would be independent of the derivative price, because the stock and derivative payoffs have a covariance of zero: $Cov\left(\tilde{x}, \tilde{V}\right) = 0$, and the risk premium in the equity market would simply equal the expected variance.

Importantly, the equity demands D_{Si} are not directly a function of investors' risk signals $\tilde{\eta}_i$, given the conjecture that $F_{P_D}\left(\cdot|P_S, \tilde{\mu}\right) = F_{P_D}\left(\cdot|P_S\right)$. Thus, despite the fact that investors' risk signals provide them with information regarding the riskiness of the stock, they choose not to take into account these signals $\tilde{\eta}_i$ when trading in the stock market. As a result, the conjecture that the stock price is informationally redundant with respect to $\tilde{V}$ is verified.

Corollary 1 *The stock price is informationally redundant with respect to $\tilde{V}$. That is, $F_{P_S}\left(\cdot|P_D, \tilde{V}\right) = F_{P_S}\left(\cdot|P_D\right)$.*

This corollary does not imply that the stock market and derivative markets function independently. Proposition 1 demonstrates that a higher derivative price P_D reduces the intensity with which investors trade on their mean information and reduces the stock price

[13] See footnote 18 in Eeckhoudt and Schlesinger (2006) for a discussion of why this holds even for CARA utility, which is generally interpreted as having a preference for risk that is independent of wealth. The notion of preferences across distributions is distinct from the Arrow-Pratt measure of risk aversion, which assesses how much an investor is willing to pay to eliminate a risk at any given wealth level. The Arrow-Pratt measure also takes into account an investor's marginal utility at a given wealth level.

P_S, as it becomes costlier to hedge the risk uncertainty induced by a position in the stock market by purchasing the derivative. Instead, the corollary only states that investors' risk information can only affect the equity price through the price of the derivative, P_D.

With the equilibrium in the stock market established, now consider the equilibrium derivative price for fixed equity demands, $\{D_{Si}\}_{i\in[0,1]}$. As the distribution of a variance must be bounded below by zero, the distribution of the payoff of the derivative cannot be assumed normal. To make the model as general as possible, we allow for an arbitrary distribution of $\tilde{V}$. In order to derive the rational expectations equilibrium in this general case, we apply the approach of Breon-Drish (2015b), summarized below.

First, conjecture a *generalized* linear equilibrium, i.e., one in which price is a monotonic transformation of a linear function of $\tilde{V}$ and $\tilde{Z}_D$. That is, start by conjecturing $P_D = \delta\left(l\left(\tilde{V}, \tilde{Z}_D\right)\right)$ where $l\left(\tilde{V}, \tilde{Z}_D\right) = a\tilde{V} + \tilde{Z}_D$ for some $a \in \Re$ to be determined as part of the equilibrium and a strictly increasing function δ. Given this conjecture, investors are able to invert $l\left(\tilde{V}, \tilde{Z}_D\right)$ from the derivative price. Due to the fact that the additive error terms in $\tilde{\eta}_i$ and $\tilde{l}$ are normally distributed, $\tilde{\eta}_i$ and $\tilde{l}$ are normally distributed conditional on $\tilde{V}$. This implies that the distribution of $\tilde{V}$ given $\left(\tilde{\eta}_i, \tilde{l}\right)$ falls into the exponential family of distributions with the following form:

$$dF_{\tilde{V}|\tilde{\eta}_i,\tilde{l}} = \exp\left\{\left(k_1(a)\,\tilde{\eta}_i + k_2(a)\,\tilde{l}\right)\tilde{V} - g\left(k_1(a)\,\tilde{\eta}_i + k_2(a)\,\tilde{l}; a\right)\right\} dH\left(\tilde{V}; a\right) \tag{5}$$

for some functions $k_1(a)$, $k_2(a)$, $g(\cdot; a)$, and $H\left(\tilde{V}, a\right)$. When investors have CARA utility and the distribution of payoffs takes this form, their demands are additively separable in their private signal $\tilde{\eta}_i$, the price signal $\tilde{l}$, and $g'^{-1}(P_D)$, where $g'^{-1}(\cdot)$ is a monotonic function. By examining the market clearing condition, it can be seen that P_D indeed takes the generalized linear form, $\delta\left(l\left(\tilde{V}, \tilde{Z}_D\right)\right)$. Moreover, the information content of price is uniquely determined by the weight that investors place on $\tilde{\eta}_i$ relative to the degree of noise trade in the economy, implying that there exists a unique generalized linear equilibrium.[14]

The unique equilibrium may be expressed explicitly, but, in general, it is not available in closed form. The next proposition summarizes these results.

Proposition 2 *Investors' derivative demands and the unique generalized linear equilibrium price P_D given equity demands $\{D_{Si}\}_{i\in[0,1]}$ satisfy:*

$$D_{Di} = \tau\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i + \frac{2\tau}{\sigma_e^2\sigma_D^2}\tilde{l}\right) - \tau g'^{-1}(P_D) + \frac{1}{2\tau}D_{Si}^2 \quad (6)$$

$$P_D = g'\left(\frac{1}{2\tau}\left[\left(1+\frac{4\tau^2}{\sigma_e^2\sigma_D^2}\right)\left(\frac{2\tau}{\sigma_e^2}\tilde{V} + \tilde{Z}_D\right) + \frac{1}{2\tau}\left(\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2\right)\right]\right) \quad (7)$$

where

$$g(\xi) \equiv \log\left[\int_\Upsilon \exp\left\{\xi v - \frac{1}{2\sigma_e^2}\left(1+\frac{\tau^2}{\sigma_e^2\sigma_D^2}\right)v^2\right\} dF_V(v)\, dv\right]$$

and $\tilde{l} \equiv \frac{2\tau}{\sigma_e^2}\tilde{V} + \tilde{Z}_D$*.* P_D *is positive and increasing in* $\tilde{V}$ *and* $\tilde{Z}_D$*.*

Critically, investors' derivative demands have a new component that arises due to the fact that this is a derivative security, revealing an important interaction between the two markets. In particular, investor i's demand is increasing in the square of their position in the stock, $\frac{1}{2\tau}D_{Si}^2$, as a result of their desire to hedge risk uncertainty. This implies that both investors who short the stock and investors who long the stock will hold positions in the derivative; henceforth, we refer to this component of an investor's demand for the derivative as the risk-uncertainty hedging demand. The derivative price depends on investors' equity demands through the aggregate hedge of informed traders, plus the hedge from noise traders in the stock market, $\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2$.

Although investors' equity demands affect the derivative price, and these demands depend upon $\tilde{\varphi}_i$ and $\tilde{Z}_S$, the derivative price is nevertheless informationally redundant with respect

[14] It can be shown that when the distribution of $\tilde{V}$ is continuous, this generalized linear equilibrium is the unique equilibrium amongst the class of equilibria in which price is continuous and satisfies $\frac{dP}{dV} \neq 0$ for almost every $V \in \Upsilon$ (see Breon-Drish (2015b) Proposition 2.2). Intuitively, even if investors were to derive a signal from price that is not linear in $\tilde{V}$ and $\tilde{Z}_D$, their demands for the asset would still be linearly separable in their private signals and the price signal.

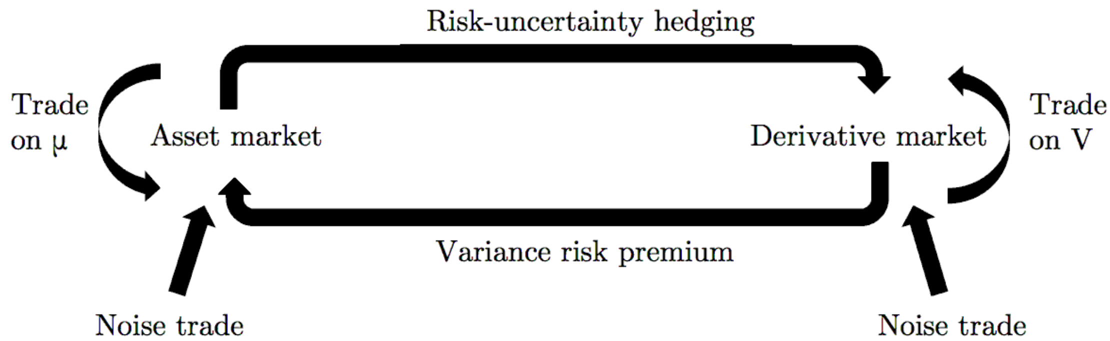

Figure 1: This figure depicts the interplay between the asset and the derivative market, given that the derivative insures against fluctuations in $\tilde{V}$ induced by a position in the asset market. Trade in the asset affects the derivative price through investors' desire to hedge risk uncertainty. This in turn, affects the asset market through the risk premium associated with $\tilde{V}^{\frac{1}{2}}\tilde{f}$.

to $\tilde{\mu}$. Substituting the equity price in Proposition 1 into investors' equity demands, we find that their equity demands are linear in the difference between their private mean signal ($\tilde{\varphi}_i$) and the average mean signal ($\tilde{\mu}$). As a result, the aggregate risk-uncertainty hedging demand of informed investors for the derivative, $\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2$, is a function only of $\tilde{\mu}$ only through the term $\int_0^1 (\tilde{\varphi}_i - \tilde{\mu})^2 di$. It is easily checked that $\int_0^1 (\tilde{\varphi}_i - \tilde{\mu})^2 di$ depends only upon investors information quality, and is unaffected by $\tilde{\mu}$ itself.[15] Hence, we have the following result:

Corollary 2 *The derivative price is informationally redundant with respect to $\tilde{\mu}$. That is, $F_{P_D}(\cdot|P_S, \tilde{\mu}) = F_{P_D}(\cdot|P_S)$.*

Now that the two markets have been examined in isolation, taking the price and demands in the other market as fixed, we consider both markets in tandem and show that there exists a unique equilibrium.

Proposition 3 *There exists a unique rational expectations equilibrium P_S, P_D.*

[15] This is somewhat of a knife-edged case that serves to make the model tractable. If investors instead had heterogenous information precisions regarding $\tilde{\mu}$, or if the noise traders in the stock market did not hedge their positions in the derivative market, the derivative price would reveal information regarding $\tilde{\mu}$ or $\tilde{Z}_A$, respectively.

The existence of a unique equilibrium boils down to a fixed point problem; the nature of this problem is depicted in Figure 1. Expression 2 shows that the aggregate risk-uncertainty hedging demand of investors, $\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2$, is a function of the derivative price, P_D. In particular, in the appendix we show that:

$$\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2 = \frac{\int_0^1 \left(E\left(\tilde{\mu}|\Phi_i\right) - \int_0^1 E\left(\tilde{\mu}|\Phi_i\right) di \right)^2 di}{\tau^{-2}\left(P_D + Var\left(\tilde{\mu}|\Phi_i\right)\right)^2}. \tag{8}$$

The numerator of this expression, which captures the variation in investors' beliefs, $\int_0^1 \left(E\left(\tilde{\mu}|\Phi_i\right) - \int_0^1 E\left(\tilde{\mu}|\Phi_i\right) di \right)^2 di$, depends upon P_D. Expression 2 reveals that the size of P_D determines the sensitivity of the stock price to noise trade, and hence, the precision of the signal derived from the stock price. Moreover, the denominator is also a direct function of P_D; intuitively, when P_D is larger, investors face more risk and are more reluctant to trade on their information. Simultaneously, the derivative price is itself a function of $\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2$, such that finding an equilibrium requires solving for a fixed point P_D. A unique solution exists to this fixed point problem. Briefly, the intuition for why this fixed point problem has a unique solution is that a higher derivative price causes a reduction in the investors' aggregate hedging demands $\int D_{Si}^2 di + \tilde{Z}_S^2$, which works against the initial increase in the derivative price. Effectively, this implies that the fixed point problem is a contraction.

Before moving on, it is insightful to compare our model to a simpler framework in which investors trade in two correlated securities, S_1 and S_2, where S_1 pays off the sum of two independent normally distributed components, $\tilde{\rho}_1 + \tilde{\rho}_2$, S_2 pays off $\tilde{\rho}_2$, and investors receive signals on both $\tilde{\rho}_1$ and $\tilde{\rho}_2$. Assume further that investors' information signals are normally distributed conditional on $\tilde{\rho}_1$ and $\tilde{\rho}_2$, and that there is independent, normally distributed noise trade in each security. This setting parallels the present one in that investors trade in two related securities, and in particular, one of these securities (S_2) can be used to hedge a portion of the risk induced by a position in the other security (S_1). Roughly, S_1 corresponds to the equity and S_2 corresponds to the derivative in our setting, $\tilde{\rho}_1$ corresponds to $\tilde{\mu}$ and

$\tilde{\rho}_2$ corresponds to $\tilde{V}^{\frac{1}{2}}\tilde{f}$. Similar to the results of propositions 1 and 2, it can be verified that investors trade on information regarding $\tilde{\rho}_1$ in S_1 and S_2, and trade on information regarding $\tilde{\rho}_2$ in S_2 only. Intuitively, investors first determine their optimal positions in S_1 based on their information regarding $\tilde{\rho}_1$, and then adjust their positions in S_2 to achieve optimal exposures to $\tilde{\rho}_2$. The investors' demands for S_1 increase linearly in the price of S_2, which captures how much it would cost the investor to replicate the exposure to $\tilde{\rho}_2$ resulting from a position in S_1 by trading in S_2.

Despite these similarities, there are important discrepancies between this simple setting and the present model that are critical to the applications we consider. In the simple framework, S_2 has a payoff that directly enters the payoff of S_1, and thus enters the expected payoff of S_1. On the other hand, the derivative pays off $\tilde{V}$, which enters the *variance* of a position in the equity. As a result, while the price of S_2 enters linearly in the demand of an investor for S_1, the price of the derivative appears in the denominator of investors' demands. Unlike the simpler setting, this leads the derivative price to affect how intensely investors trade on their mean information and causes the derivative price to enter the equity's risk premium. These forces are crucial to the results in sections 3.1, 3.3, and 3.4. Moreover, investors' demands for S_2 decrease linearly in their demands for S_1. In our model, investors' demands for the derivative increase quadratically in their equity positions, which is crucial to the results in section 3.2.

Note that allowing for a non-normally distributed payoff in the derivative market is a necessary component to examining a model with risk-based information, given that any variance distribution must be bounded below by zero. However, within such a general framework, some of the well-known results regarding the impact of uncertainty and noise trade on expected returns may no longer hold. Thus, it is important to distinguish which features of our results stem from variance uncertainty and the presence of a derivative security, rather than from the consideration of a generalized distribution. For example, it is possible that the expected returns to the derivative actually decrease in the degree of uncertainty over the derivative's payoff. This is not a consequence of the interaction between the two markets, but arises purely due to the fact that the variance is not necessarily normally distributed. To deal with this concern, we leave many of our comparative statics in terms of shocks to P_D, without specifically specifying which of the underlying parameters lead to this change. This enables us to focus on the economic forces of the interaction between the two markets, while leaving the details regarding how uncertainty affects prices in the case of a generalized distribution to future research.

3 Applications

3.1 Trading volume and price changes

In this section, we analyze the relationship between trading volume and price changes within and across the equity and derivative markets. An empirical literature has documented associations between stock and option volume and stock returns. For example, a literature surveyed by Karpoff (1987) finds that trading volume in the stock market is positively related to stock returns. Existing rational expectations models have found a relationship between absolute volume and price changes (Kim and Verrecchia (1991)) and a relationship between signed volume and price changes (Schneider (2009)). In our model, signed volume is associated with contemporaneous stock returns, but for a different reason than in prior literature.[16] Innovations to the riskiness of cash flows or noise trade in the derivative market lead to simultaneous changes in the risk premium charged by investors in the stock market and in investors' willingness to trade on their information.

Moreover, our model speaks to the empirical relationship between option volume and future stock returns (Easley, O'Hara, and Srinivas (1998), Pan and Poteshman (2006)). Prior literature argues that informed traders first take their information to the derivative

[16] The relationship between absolute price changes and trading volume in the asset market in my model is less clear. Unlike Kim and Verrecchia (1991), we assume that investors have homogenous information quality, such that revisions to the underlying security cause investors to update their beliefs equally.

market, due to the leverage provided; this leads to a positive association between volume and future stock returns as the stock market is slow to reflect the information in options trade. However, the empirical measures of volume used in this literature correspond to trade in derivatives that respond to directional movements in stock price. Our model suggests that volume in derivative positions that fluctuate primarily in the absolute movement in stock price, such as straddles or variance swaps, should be *negatively* associated with future returns. Intuitively, innovations to payoff risk or noise trade in the derivative market raise the derivative price. This reduces the stock price and decreases investors' willingness to trade on their mean information. Moreover, trading volume in the derivative falls as well, since investors' desires to hedge risk uncertainty converge as they trade less in the stock. Finally, the model predicts a relationship between derivative prices and trading volume.

We begin by characterizing trading volume in the two markets. Formally, we examine trade amongst informed investors, ignoring trade between the noise traders and informed traders,[17] and define trading volume in the stock to equal $Vol_S \equiv \int_0^1 \left| D_{Si} - \int_0^1 D_{Si} di \right| di$ and trading volume in the derivative to equal $Vol_D \equiv \int_0^1 \left| D_{Di} - \int_0^1 D_{Di} di \right| di$. The next proposition formally characterizes volume in the two markets.

Proposition 4 *Trading volume in the stock market is equal to:*

$$Vol_S = \tau \frac{\int_0^1 \left| E\left(\tilde{\mu}|\Phi_i\right) - \int_0^1 E\left(\tilde{\mu}|\Phi_i\right) di \right| di}{P_D + Var\left(\tilde{\mu}|\Phi_i\right)} \tag{9}$$

and trading volume in the derivative market is equal to:

$$Vol_D = \int_0^1 \left| \frac{\tau}{\sigma_e^2}\left(\tilde{\eta}_i - \tilde{V}\right) + \frac{1}{2\tau}\left(D_{Si}^2 - \int_0^1 D_{Si}^2 di\right) \right| di. \tag{10}$$

Volume in the stock market is proportional to the absolute dispersion of investors' beliefs

[17] Prior noisy rational expectations models which study trading volume effectively do the same by assuming that the noise in the economy takes the form of noisy supply (e.g., Kim and Verrecchia (1991)), or a random exogenous outside income (Schneider (2009), Wang (1994)).

about the stock's expected payoffs, $\int_0^1 \left| E(\tilde{\mu}|\Phi_i) - \int_0^1 E(\tilde{\mu}|\Phi_i)\,di \right| di$. Dispersion in investors' equilibrium beliefs causes them to take varied positions in the stock, leading to trade. Furthermore, investors' willingness to trade on their information is disciplined by their assessment of the riskiness of the stock's payoffs, as measured by $\tau^{-1}(P_D + Var(\tilde{\mu}|\Phi_i))$. This reveals an interaction between the stock and derivative markets: any force that raises the derivative price also leads to a reduction in trade in the stock market. For instance, a change in the distribution of $\tilde{V}$ or in investors' risk information quality that causes P_D to increase, or an increase in noise trade in the derivative will reduce trade in the stock market.

Next, Proposition 4 shows that derivative trading volume has two components, which we refer to as the risk information component of volume and risk-uncertainty hedging component of volume. The risk information component of volume is captured by the term $\frac{\tau}{\sigma_e^2}\left(\tilde{\eta}_i - \tilde{V}\right)$, while the risk-uncertainty hedging component of volume is captured by $\frac{1}{2\tau}\left(D_{Si}^2 - \int_0^1 D_{Si}^2 di\right)$. Risk information related trade is intuitive, and is driven by the deviation between the signal received by investor i, $\tilde{\eta}_i$, and the average signal received by investors, $\int_0^1 \tilde{\eta}_i di = \tilde{V}$. On the other hand, hedging related trade results from differences in investors' equilibrium beliefs regarding $\tilde{\mu}$. Such differences in beliefs lead investors to hold disparate positions in the stock. Investors who hold large positions in the stock hedge the resulting variance uncertainty by purchasing the derivative from investors with smaller positions in the stock. Expanding this component of trade yields:

$$D_{Si}^2 - \int_0^1 D_{Si}^2 di = \frac{\left(E(\tilde{\mu}|\Phi_i) - \int_0^1 E(\tilde{\mu}|\Phi_i)\,di\right)^2}{\tau^{-2}\left(P_D + Var(\tilde{\mu}|\Phi_i)\right)^2}. \tag{11}$$

This component of derivative trade thus highly resembles trade in the stock market, except that it is a squared, rather than absolute measure. As a result, forces that create trade in the stock market will also generate trade in the derivative market. Moreover, the risk-uncertainty hedging component of derivative trade is a function of P_D. Hence, the quality of investors' risk information and the extent of noise trade can affect trade in the derivative market in two fashions: directly, through the risk-information component of trade, and indirectly, through their impact on investors' willingness to trade on their mean information. This renders comparative statics with respect to private risk information difficult. Nevertheless, increases in the derivative price driven by $\tilde{V}$ or $\tilde{Z}_D$ affect the risk-uncertainty hedging component of trading volume only, and hence, definitively lead to a reduction in derivative volume. In sum, we have the following corollary.

Corollary 3 *i) Increases in the derivative price cause a decline in trade in the stock market. ii) Increases in the derivative price driven by $\tilde{V}$ or $\tilde{Z}_D$ cause a decline in trade in the derivative market.*

With this foundation in mind, we consider the covariance between volume in the two markets and price changes in the two markets. Mathematically, this involves considering the direction of the impact of each of the underlying random variates, $\tilde{\mu}$, $\tilde{V}$, $\tilde{Z}_S$, and $\tilde{Z}_D$, on prices and volume. Formally, suppose that prior to trade, there exist initial prices P_{S0} and P_{D0} and define $\Delta P_S \equiv P_S - P_{S0}$ and $\Delta P_D \equiv P_D - P_{D0}$. Then, we have the following corollary.

Corollary 4 *The statistical relationship between contemporaneous returns in the stock and derivative and trading volume in the two markets can be summarized as follows:*
i) $Cov\left(\Delta P_S, Vol_S\right) > 0$*,*
ii) $Cov\left(\Delta P_S, Vol_D\right) > 0$*,*
iii) $Cov\left(\Delta P_D, Vol_S\right) < 0$*,*
iv) $Cov\left(\Delta P_D, Vol_D\right) < 0$*.*

The corollary may be understood by considering the relationship between the risk premium in the stock market, the derivative price, and trading volume in the two markets. Proposition 1 reveals two key predictions: i) forces that drive up the derivative price increase the risk premium in the stock market, and ii) increases in the risk premium reduce investors' willingness to trade on their mean-based information. Specifically, innovations in the stochastic variance $\tilde{V}$ or noise trade in the derivative $\tilde{Z}_D$ increase the derivative price, causing an increase in the risk premium in the stock market and disciplining investors' willingness to trade on their information. Hence, price decreases in the stock market are associated with heightened derivative prices, decreased trade in the stock, and decreased trade in the derivative market.

Finally, note that our results have implications for the empirical literature that studies option trading volume relative to equity trading volume (or O:E) (Roll, Schwartz, and Subramanyam (2010), Johnson and So (2012)). While private mean information leads investors to trade in both the equity and derivative, risk information leads investors to trade in the derivative only. Intuitively, this might suggest that the O:E ratio should decrease in the amount of private risk information, but may increase or decrease in the amount of private mean information. However, our results suggest that an additional force is present - when investors have more precise risk information, the derivative price itself changes, which can lead investors to trade more intensely on their mean information. Consequently, the net effect of mean and risk information quality on the O:E ratio is unclear.

3.2 Belief Dispersion and Prices

A well-documented result is that in a perfectly competitive rational expectations equilibrium, only the average expectation of payoffs across investors and the average precision of investors' beliefs affect expected returns (e.g., Banerjee (2011), Lambert, Leuz, and Verrecchia (2012)). The thought experiment posed by these studies is as follows. Consider a change in the underlying parameters of the model, such as the quality of investors' private information and the extent of noise trade, that causes investors' beliefs and equilibrium demands to diverge, but leads to no change in the average quality of their information. These studies show that when investors have CARA utility, competition is perfect, and the stock's payoff is normally distributed, the risk premium is purely a function of the precision of investors' posteriors, and hence, prices will not change, on average. Nevertheless, empirical research has shown that disagreement amongst investors is associated with lower future returns (Diether, Malloy, and Scherbina (2002), Goetzmann and Massa (2005)). Our model suggests that the effect of disagreement on a stock's price will be moderated by the extent of uncertainty regarding the riskiness of its payoffs.

Expression 6 demonstrates that the derivative price, and hence the risk premium in the stock market, increase in the aggregate *squared* demands of investors, plus the squared demand of noise traders, $\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2$. This term reflects the aggregate desire of investors to hedge the risk uncertainty created by their positions in the stock. Upon simplifying $\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2$, we find that

$$\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2 = \tau^2 \frac{\int_0^1 \left(E\left(\tilde{\mu}|\Phi_i\right) - \int_0^1 E\left(\tilde{\mu}|\Phi_i\right) di \right)^2 di}{\left(P_D + Var\left(\tilde{\mu}|\Phi_i\right)\right)^2}, \tag{12}$$

i.e., $\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2$ is a function of the dispersion in informed investors' beliefs as defined in prior literature (e.g., Banerjee (2011)), $\int_0^1 \left(E\left(\tilde{x}|\Phi_i\right) - \int_0^1 E\left(\tilde{x}|\Phi_i\right) di \right)^2 di$, weighted by investors' squared perception of the stock's risk, $\left(P_D + Var\left(\tilde{\mu}|\Phi_i\right)\right)^2$. Intuitively, when investors' beliefs grow disparate, optimistic investors increase their positions in the stock market, and pessimistic investors decrease their positions. Investors require a risk premium as compensation for the risk induced by their positions. In the standard model, the increased risk premium charged by optimistic investors is exactly offset by the decreased risk premium charged by pessimistic investors. However, in the face of uncertain risk, the compensation that an investor charges to hold an equity position grows more rapidly in the size of this position. This causes the equity risk premium to rise in dispersion in investors' beliefs. Moreover, it implies that dispersion in beliefs leads to a greater aggregate hedge in the derivative market, and a higher derivative price.

With this in mind, it is interesting to consider the forces in the model that lead investors' beliefs to diverge. First, and trivially, exogenous variation in investors' prior beliefs about the expected equity payoff creates belief dispersion. So far, we have assumed that investors have homogenous prior beliefs regarding $E(\tilde{\mu})$ equal to m. When investors have heterogenous prior beliefs over $E(\tilde{\mu})$, determined by a measurable function $m_i : [0,1] \longmapsto \Re$, we have the following result:

Proposition 5 *The derivative price P_D increases in $\int_0^1 \left(m_i - \int_0^1 m_i di\right)^2 di$. As a result, the expected stock price $E(P_S)$ decreases in $\int_0^1 \left(m_i - \int_0^1 m_i di\right)^2 di$.*

This proposition suggests that an increase in belief dispersion causes a drop in the stock price, contradicting the empirical findings of Diether et al. (2002) and Goetzmann and Massa (2005). However, this result is predicated upon the absence short sale constraints and limited liability, both of which have been identified as drivers of the relationship between disagreement and prices (Diether et al. (2002), Johnson (2004)). Nevertheless, the result suggests that the higher levels of risk uncertainty attenuate the positive relationship between stock prices and belief dispersion.

Next, we study how investors' prior uncertainty regarding $\tilde{\mu}$, the quality of their information regarding $\tilde{\mu}$, and noise trade in the stock market affect belief dispersion, and their resulting impact on the derivative and stock prices. First, note that belief dispersion is non-monotonic in the quality of investors' information regarding $\tilde{\mu}$, σ_ε^2. Hence, the derivative price is non-monotonic in σ_ε^2. Intuitively, when investors' information is very noisy, they pay little attention to their private signals, and hence, share relatively similar beliefs. On the other hand, when investors' information is very precise, they heavily weight on their private signals, but these signals are very accurate, and hence very similar.

On the other hand, belief dispersion increases monotonically in investors' prior uncertainty regarding $\tilde{\mu}$, σ_μ^2, and the amount of noise trade in the stock market, σ_S^2. Weaker priors cause investors to place more weight on their private information, and less weight on their common prior. Likewise, a higher level of noise trade leads investors to place more weight on their private information and less weight on the common signal that they receive from price. Consequently, the derivative price increases in σ_μ^2 and σ_S^2.

The pricing of belief dispersion has implications for how σ_ε^2, σ_μ^2, and σ_S^2 affect the expected stock price. Traditional noisy rational expectations models suggest that higher quality private information reduces stocks' expected returns by leading to a reduction in investors' perceived uncertainty. On the other hand, in our model, this can be offset by the effect of σ_ε^2 on the degree of dispersion in investors' equilibrium beliefs. Formally, applying the chain rule, we can write:

$$\frac{dE\left(P_S\right)}{d\sigma_\varepsilon^2} \propto -\underbrace{\frac{dE\left[Var\left(\tilde{\mu}|\Phi_i\right)\right]}{d\sigma_\varepsilon^2}}_{\text{change in belief precision}} - \underbrace{E\left(\frac{dP_D}{d\int_0^1 D_{Si}^2 di}\frac{d\int_0^1 D_{Si}^2 di}{d\sigma_\varepsilon^2}\right)}_{\text{change in belief dispersion}}. \tag{13}$$

More precise private information has two effects: first, it has the standard effect of reducing investors' uncertainty regarding $\tilde{\mu}$, which is captured by $\frac{dE[Var(\tilde{\mu}|\Phi_i)]}{d\sigma_\varepsilon^2}$. Second, more precise private information increases belief dispersion, causing the derivative price to rise, and making it costlier to hedge risk uncertainty. Depending upon the precise parameters of the model, it is feasible that the stock's expected price is non-monotonic in the quality of investor information. As an illustrative example, in figure 2, we plot the expected stock price as a function of σ_ε^2 in the case in which the variance distribution is binary, and, for simplicity, investors possess only mean information. Given the chosen parameters, the effect of σ_ε^2 on $Var\left(\tilde{\mu}|\Phi_i\right)$ is immaterial relative to the effect on P_D, such that the stock's risk premium rises in σ_ε^2.

Finally, note that increases in σ_μ^2 and σ_S^2 not only decrease the expected stock price by increasing uncertainty, but also increase belief dispersion, amplifying their negative impact on the expected stock price. Formally, $\frac{dE(P_S)}{d\sigma_\mu^2} < -\tau^{-1}\frac{\partial Var(\tilde{\mu}|\Phi_i)}{d\sigma_\mu^2}$ and $\frac{dE(P_S)}{d\sigma_S^2} < -\tau^{-1}\frac{\partial Var(\tilde{\mu}|\Phi_i)}{d\sigma_S^2}$. The next corollary summarizes these results.

Corollary 5 *i) The derivative price is non-monotonic in investors' private information quality regarding $\tilde{\mu}$, σ_ε^2, increasing in investors' prior uncertainty regarding $\tilde{\mu}$, σ_μ^2, and increasing in the variance of noise trade in the stock market, σ_S^2.*

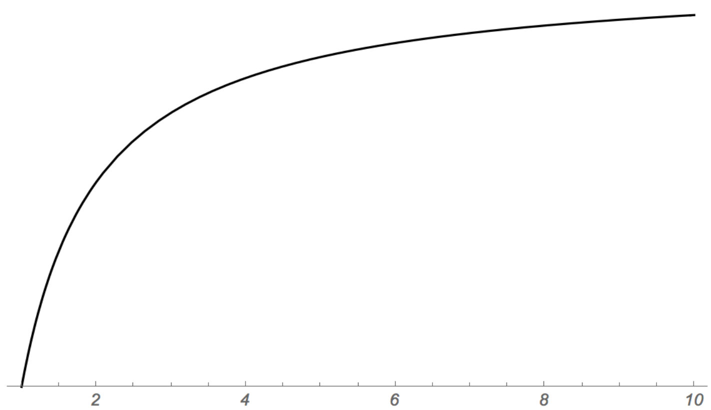

Figure 2: This picture plots the asset price as a function of σ_ε^2 when $\tilde{V} = 2$ with probability $\frac{1}{2}$ and 0 with probability $\frac{1}{2}$. In the simulation, investors possess only mean information, $\sigma_A^2 = 0.10$, $\sigma_\mu^2 = 0.01$, $m = 0$, and $\tau = 5$. Notably, the asset price increases in σ_ε^2.

ii) The effect of private information quality regarding $\tilde{\mu}$, σ_ε^2, on the expected stock price is ambiguous. The negative effect of σ_μ^2 and σ_S^2 on the expected stock price is compounded by risk uncertainty.

3.3 Risk Uncertainty and Investor Learning

In this section, we study how the presence of risk uncertainty impacts investors' ability to learn information regarding expected stock payoffs from the equity price. Our results suggest that greater derivative prices reduce the efficiency of the stock price with respect to investors' private information. Moreover, we show that investors' desire to hedge risk uncertainty amplifies the effect of noise trader shocks in the derivative market on derivative prices.

Expression 2 reveals that the amount of information that investors are able to derive from the stock price is a function of P_D, and is thus stochastic. The realized variance $\tilde{V}$, noise trade in the derivative market $\tilde{Z}_D$, and belief dispersion increase the derivative price, P_D. This raises the cost of hedging risk uncertainty induced by a position in the stock, causing the stock's risk premium, $\tau^{-1}\tilde{Z}_S\left(P_D + Var\left(\tilde{\mu}|\Phi_i\right)\right)$, to rise in magnitude, for any realized noise trader demand, $\tilde{Z}_S$. This makes the stock price more sensitive to noise trade

$\tilde{Z}_S$ relative to the investors' private information regarding $\tilde{\mu}$, reducing investors' ability to learn relevant information from the stock price. As a result, investors face greater residual uncertainty over $\tilde{\mu}$ as $\tilde{V}$, $\tilde{Z}_D$, or belief dispersion rise. Conversely, noise trader sales in the derivative market reduce the risk premium in the stock market and enhance learning, by effectively offering informed investors a free hedge in the derivative market. This increases the amount that investors learn from price.

Define the efficiency of the stock price to equal the inverse variation in the stock price for a fixed level $\tilde{x}$, $Var\left(P_S|\tilde{x}\right)^{-1}$. Then, we have the following corollary.

Corollary 6 *Investors' residual uncertainty regarding $\tilde{\mu}$, $Var\left(\tilde{\mu}|\Phi_i\right)$, and the efficiency of the stock price, $Var\left(P_S|\tilde{x}\right)^{-1}$ decrease in noise trade in the derivative market, $\tilde{Z}_D$, in the uncertain variance $\tilde{V}$, and in belief dispersion $\int_0^1 \left(m_i - \int_0^1 m_i di\right)^2 di$.*

Finally, note that the interaction between the two markets amplifies the effect of noise trade and the realized variance on the derivative price. As $\tilde{V}$ and $\tilde{Z}_D$ reduce learning from prices, they lead to greater dispersion in investors' equilibrium beliefs. This leads investors' aggregate risk-uncertainty hedging demand in the derivative to rise, causing P_D to rise further.

3.4 Variance Risk Premium

A large body of empirical evidence has demonstrated that the pricing of a security that pays off equal to the realized variance of returns (i.e., a variance swap) overshoots investors' expectations of the future variance, and has termed the difference between these two quantities the variance risk premium (e.g., Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009)). Existing theory typically explains this phenomenon as resulting from variance swaps' negative betas (Carr and Wu (2009)) or from a preference for higher moments (Bakshi and Madan (2006)). In line with the latter explanation, in our setting, a variance risk premium can arise due to investors' preference for skewness; the derivative pays off in states in risky states when investors' marginal utility tends to be high. Define the variance risk premium as -1 times the average investor's expectation of future returns to the derivative, i.e.,

$$VRP \equiv P_D - \int_0^1 E\left(\tilde{V}|\Phi_i\right) di. \tag{14}$$

This definition captures the difference between how investors price the variance $\tilde{V}$, and the average expectation of the future variance. An advantage to studying the variance risk premium in a model of heterogeneously informed investors is that it can be related to trading volume and investors' private information quality. In particular, the variance risk premium increases in the size of investors' risk-uncertainty hedging demands in the derivative security, which are a function of belief dispersion and the distribution of $\tilde{\mu}$ and $\tilde{\varphi}_i$. By applying the results from Corollary 5 and Proposition 5, the variance risk premium increases in belief dispersion $\int_0^1 \left(m_i - \int_0^1 m_i di\right)^2$, is non-monotonic in σ_ε^2, increases in σ_μ^2, and increases in σ_S^2.

Moreover, the model suggests that the variance risk premium is negatively associated with trading volume in the two markets. This is a direct consequence of Corollary 4: when the variance risk premium is high, investors are more averse to the risk uncertainty that results from a position in the stock market. This makes them less willing to trade on their information regarding $\tilde{\mu}$, such that trading volumes in both the stock and derivative are lower. Additionally, empirical evidence suggests that derivative returns and the returns of their underlyings often move in opposite directions (Bakshi, Cao, and Chen (2000)), and that the variance risk premium predicts future equity returns (Bollerslev, Tauchen, and Zhou (2009)). In the present setting, there exists a direct relationship between the size of the variance risk premium and stock returns. This follows trivially from expression 2, which shows that the risk premium in the stock market increases with P_D. In summary, we have the following proposition.

Proposition 6 *i) The variance risk premium increases in* σ_m^2, σ_S^2, *and* σ_μ^2, *and is non-monotonic in* σ_ε^2.

ii) The variance risk premium is negatively correlated with trading volume in the stock and derivative markets.

iii) Stock returns and the variance risk premium are linearly related.

4 Conclusion

This article develops a noisy rational expectations model in which risk-averse investors possess information not only on a stock's expected payoffs, but also the risk of these payoffs. Investors in the model can trade in a stock or a derivative security whose value increases in the riskiness of the stock's payoffs. In the equilibrium studied in the model, the stock price serves as an aggregator of investors' mean information, and the derivative price serves as an aggregator of investors' risk information. Investors trade on mean information in both the stock and derivative markets, as the derivative serves as insurance against adverse fluctuations in the risk of the stock's payoffs. On the other hand, investors trade on risk information in the derivative only. The model has implications for relationship between trading volume in stock and derivative markets and the respective prices in these two markets. Moreover, it suggests that belief dispersion impacts expected stock returns and derivative prices. It demonstrates that greater derivative prices are associated with reduced price efficiency in the stock market. Finally, it justifies the empirically documented negative relationship between the variance risk premium and returns in the stock market, and offers predictions on the association between trading volume, information quality, and variance risk premia.

In the current set up of the model, investors have homogenous information quality, which leads to a derivative price that is uninformative regarding investors' information regarding expected equity payoffs. Preliminary investigation suggests that this will not be the case when investors have heterogenous information quality, since, in this case, the dispersion in their beliefs will be a function of the fundamental $\tilde{\mu}$. It may be interesting, but technically challenging, to study the value of the derivative price to investors when it aggregates both information on the mean of future cash flows and their risk. Furthermore, a weakness of the model is that because the variance distribution is fully general, it is difficult to offer much intuition into how the parameters of the variance distribution impact the derivative price. It may be interesting to study more specific distributions of the variance $\tilde{V}$ in order to offer more definitive comparative statics on the drivers of the derivative price in the model.

5 Appendix

Proof of Proposition (1). Under the conjecture that $F_{P_D}(\cdot|P_S,\tilde{\mu}) = F_{P_D}(\cdot|P_S)$, the derivative price serves no role in updating on $\tilde{\mu}$, and hence, its distribution is irrelevant in determining the posterior distribution of $\tilde{x}$ given the investors' information. Consequently, upon conditioning on the uncertain variance $\tilde{V}$, the investors' belief regarding $\tilde{x}$ is normally distributed: $\tilde{x}|\tilde{V},\Phi_i \sim N\left(E(\tilde{x}|\Phi_i), Var(\tilde{\mu}|\Phi_i)+\tilde{V}\right)$. Thus, the first order condition with respect to the equity demand and derivative demand is equal to:

$$\begin{aligned}
0 &= \frac{\partial}{\partial D_{Si}} E_V\left(-\exp\left[-\tau^{-1}D_{Si}\left(E(\tilde{x}|\Phi_i)-P_S\right)+\frac{1}{2}\tau^{-2}D_{Si}^2 Var(\tilde{\mu}|\Phi_i)\right.\right. \\
&\quad \left.\left. +\tau^{-1}D_{Di}P_D+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right]|\Phi_i\right) \\
0 &= \frac{\partial}{\partial D_{Di}} E_V\left[-\exp\left(-\tau^{-1}D_{Si}\left(E(\tilde{x}|\Phi_i)-P_S\right)+\frac{\tau^{-2}}{2}D_{Si}^2 Var(\tilde{\mu}|\Phi_i)\right.\right. \\
&\quad \left.\left. +\tau^{-1}D_{Di}P_D+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)|\Phi_i\right].
\end{aligned}$$

As we show in the proof of Proposition *2*, $\tilde{V}|\Phi_i$ lies in the exponential family, and has a moment generating function that is defined on the reals, i.e., $\forall t \in \Re,\ E\left(-\exp\left(t\tilde{V}\right)|\Phi_i\right) < \infty$. This implies that the order of differentiation and expectation can be interchanged in these two conditions,[18] which yields the following:

$$
\begin{aligned}
0 &= E_V\left(-\left(-\tau^{-1}\left(E\left(\tilde{x}|\Phi_i\right)-P_S\right)+\tau^{-2}D_{Si}\left(\tilde{V}+Var\left(\tilde{\mu}|\Phi_i\right)\right)\right)\exp\left[-\tau^{-1}D_{Si}\left(E\left(\tilde{x}|\Phi_i\right)-P_S\right)+\right.\right.\\
&\quad \left.\left.*\tau^{-2}D_{Si}^2 Var\left(\tilde{\mu}|\Phi_i\right)+\tau^{-1}D_{Di}P_D+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right]|\Phi_i\right)\\
0 &= E_V\left[-\left(\tau^{-1}P_D-\tau^{-1}\tilde{V}\right)\exp\left[-\tau^{-1}D_{Si}\left(E\left(\tilde{x}|\Phi_i\right)-P_S\right)+\right.\right.\\
&\quad \left.\left.*\tau^{-2}D_{Si}^2 Var\left(\tilde{\mu}|\Phi_i\right)+\tau^{-1}D_{Di}P_D+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}}{2}-D_{Di}\right)\tilde{V}\right]|\Phi_i\right).
\end{aligned}
$$

Dividing the two conditions by the constant functions of $\tilde{V}$ yields the following two simplified conditions:[19]

$$
\begin{aligned}
E_V\left(\left(-\frac{\tau}{D_{Si}}\left(E\left(\tilde{x}|\Phi_i\right)-P_S\right)+Var\left(\tilde{\mu}|\Phi_i\right)+\tilde{V}\right)\exp\left(\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)|\Phi_i\right) &= 0\\
E_V\left(\left(-P_D+\tilde{V}\right)\exp\left(\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)|\Phi_i\right) &= 0.
\end{aligned}
$$

Critically, note that $-\frac{\tau}{D_{Si}}\left(E\left(\tilde{x}|\Phi_i\right)-P_S\right)+Var\left(\tilde{\mu}|\Phi_i\right)$ and P_D are known constants conditional on Φ_i. Moreover, there is a unique constant k that solves:

$$
E_V\left(\left(k+\tilde{V}\right)\exp\left(\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)|\Phi_i\right)=0.
$$

---

[18] In particular, let $M(t)\equiv E\left(-\exp\left(t\tilde{V}\right)|\Phi_i\right)$. We wish to show that $\frac{d}{dt}M(t)=E\left(-\tilde{V}\exp\left(t\tilde{V}\right)|\Phi_i\right)$. In order to do so, we apply the dominated convergence theorem. We show that, for any sequence $\{t_n\}$ with $t_n\to t$, there exists a function $\kappa(V)$ such that:

$$
\left|\frac{-\exp\left(t_nV\right)-\exp\left(tV\right)}{t_n-t}\right|<\kappa(V)
$$

and $E\left(\kappa(V)\right)<\infty$. To find such a $\kappa(V)$, note that the mean value theorem implies that there exists a ξ between t_n and t such that:

$$
\begin{aligned}
\left|\frac{-\exp\left(t_nV\right)-\exp\left(tV\right)}{t_n-t}\right| &= \left|-\xi V\exp\left(\xi V\right)\right|\\
&= \xi V\exp\left(\xi V\right)
\end{aligned}
$$

Now, using the fact that $V\le\sum_{j=0}^{\infty}\frac{1}{j!}V^j=\exp(V)$, we get that $\xi V\exp\left(\xi V\right)<\exp\left(\left(\xi+1\right)V\right)$. Letting $\kappa(V)=\exp\left(\left(\xi+1\right)V\right)$, and using the fact that the MGF exists for all reals, we have the result.

[19] Since we may change the order of differentiation and expectation, and the utility function is concave, the second order condition holds.

To see this, note that:

$$\begin{aligned}
\lim_{k\to-\infty} E_V\left(\left(k+\tilde{V}\right)\exp\left(\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)|\Phi_i\right) &= -\infty \\
\lim_{k\to\infty} E_V\left(\left(k+\tilde{V}\right)\exp\left(\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)|\Phi_i\right) &= \infty
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial}{\partial k} E_V\left(\left(k+\tilde{V}\right)\exp\left(\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)|\Phi_i\right) \\
= \; & E_V\left(\exp\left(\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)|\Phi_i\right) > 0.
\end{aligned}$$

Together with the above two conditions, this implies that

$$-\frac{\tau}{D_{Si}}\left(E\left(\tilde{x}|\Phi_i\right)-P_S\right)+Var\left(\tilde{\mu}|\Phi_i\right)=-P_D,$$

or, solving for D_{Si},

$$D_{Si}=\tau\frac{E\left(\tilde{x}|\Phi_i\right)-P_S}{P_D+Var\left(\tilde{\mu}|\Phi_i\right)}.$$

The condition for market clearing requires that:

$$\begin{aligned}
\int_0^1 D_{Si}di &= -\tilde{Z}_S \\
\int_0^1 \tau\frac{E\left(\tilde{x}|\Phi_i\right)-P_S}{P_D+Var\left(\tilde{\mu}|\Phi_i\right)}di &= -\tilde{Z}_S,
\end{aligned}$$

that is,

$$P_S=\int_0^1 E\left(\tilde{x}|\Phi_i\right)di+\tau^{-1}\tilde{Z}_S\left(P_D+Var\left(\tilde{\mu}|\Phi_i\right)\right).$$

In the proof of Proposition 3, we show that there is a unique linear equilibrium price that satisfies this equation. ■

Proof of Corollary (1).

Note that there is no dependence of $E\left(\tilde{x}|\Phi_i\right)$ on $\{\tilde{\eta}_i\}_{i\in[0,1]}$, and $\tau^{-1}\tilde{Z}_S\left(P_D+Var\left(\tilde{\mu}|\Phi_i\right)\right)$ depends upon $\tilde{V}$ only through P_D. Thus, we have that $F_{P_S}\left(\cdot|P_D,\tilde{V}\right)=F_{P_S}\left(\cdot|P_D\right)$. ■

Proof of Proposition (2). We start by conjecturing a generalized linear equilibrium, as in Breon-

Drish (2015b). In such an equilibrium, we conjecture that price satisfies:

$$\begin{aligned} P_D\left(\tilde{V}, \tilde{Z}_D\right) &= \delta\left(l\left(\tilde{V}, \tilde{Z}_D\right)\right) \\ \text{where } \delta' &> 0 \\ \text{and } l\left(\tilde{V}, \tilde{Z}_D\right) &= \tilde{V} + a\tilde{Z}_D, \end{aligned}$$

for some constant a that will be determined as part of the equilibrium. Moreover, given that $F_{P_S}\left(\cdot|P_D, \tilde{V}\right) = F_{P_S}\left(\cdot|P_D\right)$, we have that $F_{\tilde{V}|P_D,P_S,\tilde{\eta}_i} = F_{\tilde{V}|P_D,\tilde{\eta}_i}$. As $\delta' > 0$, investors can invert the linear statistic $\tilde{V} + a\tilde{Z}_D$ from price, and hence, the information in P_D is equivalent to $\tilde{l}\left(\tilde{V}, \tilde{Z}_D\right)$. Thus, $F_{\tilde{V}|P_D,\tilde{\eta}_i} = F_{\tilde{V}|\tilde{l},\tilde{\eta}_i}$. Conditional on $\tilde{l}$ and $\tilde{\eta}_i$, we have that the variance distribution satisfies:

$$\begin{aligned} dF_{\tilde{V}|\tilde{l},\tilde{\eta}_i}(v) &\propto dF_V(v)\frac{1}{\sqrt{2\pi\sigma_e^2}}\exp\left\{-\frac{1}{2\sigma_e^2}\left(\tilde{\eta}_i - v\right)^2\right\}d\eta_i\frac{1}{\sqrt{2\pi\sigma_D^2}}\exp\left\{-\frac{1}{2\sigma_D^2}\left(\tilde{l} - av\right)^2\right\}dl \\ &\propto \exp\left\{\left(-\frac{a^2}{2\sigma_D^2} - \frac{1}{2\sigma_e^2}\right)v^2 + \left(\frac{1}{\sigma_e^2}\tilde{\eta}_i + \frac{a}{\sigma_D^2}\tilde{l}\right)v\right\}dF_V(v) \end{aligned}$$

such that we may write:

$$dF_{\tilde{V}|\tilde{l},\tilde{\eta}_i}(v) = \frac{\exp\left\{\left(-\frac{a^2}{2\sigma_D^2} - \frac{1}{2\sigma_e^2}\right)v^2 + \left(\frac{1}{\sigma_e^2}\tilde{\eta}_i + \frac{a}{\sigma_D^2}\tilde{l}\right)v\right\}dF_V(v)}{\int_\Upsilon \exp\left\{\left(-\frac{a^2}{2\sigma_D^2} - \frac{1}{2\sigma_e^2}\right)v^2 + \left(\frac{1}{\sigma_e^2}\tilde{\eta}_i + \frac{a}{\sigma_D^2}\tilde{l}\right)v\right\}dF_V(v)\,dv}.$$

This distribution belongs to the exponential family, i.e., it may be written in the form $\exp\left\{\left(k_1(a)\tilde{\eta}_i + k_2(a)\tilde{l}\right)v - g\left(k_1(a)\tilde{\eta}_i + k_2(a)\tilde{l}; a\right)\right\}H(v;a)$ where:

$$\begin{aligned} k_1(a) &= \frac{1}{\sigma_e^2} \\ k_2(a) &= \frac{a}{\sigma_D^2} \\ g(\xi) &= \log\left[\int_\Upsilon \exp\left\{\xi v - \frac{1}{2}\left(\frac{1}{\sigma_e^2} + \frac{a^2}{\sigma_D^2}\right)v^2\right\}dF_V(v)\right] \text{ and} \\ H(v;a) &= \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_e^2} + \frac{a^2}{\sigma_D^2}\right)v^2\right\}dF_V(v). \end{aligned}$$

This implies that $\tilde{V}|\tilde{\eta}_i, \tilde{l}$ has moment generating function:

$$E\left[\exp\left(t\tilde{V}\right)|\tilde{\eta}_i, \tilde{l}\right] = \exp\left\{g\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i + \frac{a}{\sigma_D^2}\tilde{l} + t\right) - g\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i + \frac{a}{\sigma_D^2}\tilde{l}\right)\right\}.$$

Then, the investors' optimization problem can be written:

$$\begin{aligned}
& \arg\max_{D_{Di}} E_V\left(-\exp\left(-\tau^{-1}D_{Si}\left(E\left(\tilde{x}|\Phi_i\right)-P_S\right)+\frac{\tau^{-2}}{2}D_{Si}^2 Var\left(\tilde{\mu}|\Phi_i\right)\right.\right. \\
& \left.\left.+\tau^{-1}D_{Di}P_D+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)\right) \\
= \; & \arg\max_{D_{Di}} E_V\left(-\exp\left(\tau^{-1}D_{Di}P_D+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)\right) \\
= \; & \arg\max_{D_{Di}} -\exp\left(\tau^{-1}D_{Di}P_D\right)E_V\left(\exp\left(\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\tilde{V}\right)\right) \\
= \; & \arg\max_{D_{Di}} -\exp\left(\tau^{-1}D_{Di}P_D\right)\exp\left\{g\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\right)-g\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}\right)\right\} \\
= \; & \arg\max_{D_{Di}} -\exp\left\{g\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\right)-g\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}\right)+\tau^{-1}D_{Di}P_D\right\}.
\end{aligned}$$

Taking the first order condition yields:

$$\begin{aligned}
0 &= \frac{\partial}{\partial D_{Di}}\left[g\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\right)\right. \\
&\quad \left.-g\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}\right)+\tau^{-1}D_{Di}P_D\right] \\
0 &= -g'\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}+\tau^{-1}\left(\tau^{-1}\frac{D_{Si}^2}{2}-D_{Di}\right)\right)+P_D \\
D_{Di} &= \tau\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}\right)-\tau g'^{-1}\left(P_D\right)+\frac{1}{2\tau}D_{Si}^2.
\end{aligned}$$

Note that noise traders in the stock market have identical derivative demands, except that their equity demands are replaced by $\tilde{Z}_S$; hence, their demands for the derivative market take the same form given their signals $\tilde{\eta}_{Zi}$. Hence, the market clearing condition is:

$$\int_0^1\left[\tau\left(\frac{1}{\sigma_e^2}\tilde{\eta}_i+\frac{a}{\sigma_D^2}\tilde{l}\right)-\tau g'^{-1}\left(P_D\right)+\frac{1}{2\tau}D_{Si}^2\right]di+\int_0^1\left[\tau\left(\frac{1}{\sigma_e^2}\tilde{\eta}_{Zi}+\frac{a}{\sigma_D^2}\tilde{l}\right)-\tau g'^{-1}\left(P_D\right)+\frac{1}{2\tau}\tilde{Z}_S^2\right]di+\tilde{Z}_D=0.$$

Applying the law of large numbers and simplifying yields:

$$P_D=g'\left(\frac{1}{2\tau}\left[2\tau\left(\frac{1}{\sigma_e^2}\tilde{V}+\frac{a}{\sigma_D^2}\tilde{l}\right)+\tilde{Z}_D+\frac{1}{2\tau}\left(\int_0^1 D_{Si}^2 di+\tilde{Z}_S^2\right)\right]\right).$$

In order for this to satisfy our conjecture that price depends on $\tilde{V}$ and $\tilde{Z}_D$ only through the linear statistic $l\left(\tilde{V},\tilde{Z}_D\right)$, it must be the case that:

$$\tilde{l}=\frac{2\tau}{\sigma_e^2}\tilde{V}+\tilde{Z}_D$$

which implies that there is a unique generalized linear equilibrium: $a=\frac{2\tau}{\sigma_e^2}$. ■

Proof of Corollary (2). Expression (6) shows the only dependence of P_D on $\tilde{\mu}$ is through the term, $\int D_{Si}^2 di - \tilde{Z}_S^2$. Moreover, note that:

$$\begin{aligned}
\int_0^1 D_{Si}^2 di - \tilde{Z}_S^2 &= \int_0^1 \left(\frac{E(\tilde{x}|\Phi_i) - \int_0^1 E(\tilde{x}|\Phi_i)\, di - \tau^{-1}\tilde{Z}_S (P_D + Var(\tilde{\mu}|\Phi_i))}{\tau^{-1}(P_D + Var(\tilde{\mu}|\Phi_i))} \right)^2 di - \tilde{Z}_S^2 \\
&= \int_0^1 \left(\frac{E(\tilde{x}|\Phi_i) - \int_0^1 E(\tilde{x}|\Phi_i)\, di}{\tau^{-1}(P_D + Var(\tilde{\mu}|\Phi_i))} - \tilde{Z}_S \right)^2 di - \tilde{Z}_S^2 \\
&= \int_0^1 \left(\frac{E(\tilde{x}|\Phi_i) - \int_0^1 E(\tilde{x}|\Phi_i)\, di}{\tau^{-1}(P_D + Var(\tilde{\mu}|\Phi_i))} \right)^2 di \\
&\quad + \frac{2\tilde{Z}_S}{\tau^{-1}P_D} \int_0^1 \left(E(\tilde{x}|\Phi_i) - \int_0^1 E(\tilde{x}|\Phi_i)\, di \right) di + \tilde{Z}_S^2 - \tilde{Z}_S^2 \\
&= \int_0^1 \left(\frac{E(\tilde{x}|\Phi_i) - \int_0^1 E(\tilde{x}|\Phi_i)\, di}{\tau^{-1}(P_D + Var(\tilde{\mu}|\Phi_i))} \right)^2 di.
\end{aligned}$$

Finally, note that this term does not depend on $\tilde{\mu}$ since:

$$\begin{aligned}
& \int_0^1 \left(E(\tilde{x}|\Phi_i) - \int_0^1 E(\tilde{x}|\Phi_i)\, di \right)^2 di \\
= & \int_0^1 \left(\left(\frac{\sigma_\varepsilon^{-2}}{h^{-2}\sigma_S^{-2} + \sigma_\mu^{-2} + \sigma_\varepsilon^{-2}} \right) (\tilde{\varphi}_i - \tilde{\mu}) \right)^2 di \\
= & \left(\frac{\sigma_\varepsilon^{-2}}{h^{-2}\sigma_S^{-2} + \sigma_\mu^{-2} + \sigma_\varepsilon^{-2}} \right)^2 \int_0^1 \tilde{\varepsilon}_i^2 di,
\end{aligned}$$

and hence the conjecture has been verified. ■

Proof of Proposition (3). Using the results from Proposition (1) and Proposition (2), a rational expectations equilibrium must simultaneously satisfy the following three conditions:

$$\begin{aligned}
(1)\ D_{Si} &= \frac{E(\tilde{x}|\Phi_i) - P_S}{\tau^{-1}(P_D + Var(\tilde{\mu}|\Phi_i))} \\
(2)\ P_S\left(\tilde{\mu}, \tilde{Z}_S\right) &= \int_0^1 E(\tilde{x}|\Phi_i)\, di + \tau^{-1}\tilde{Z}_S (P_D + Var(\tilde{\mu}|\Phi_i)) \\
(3)\ P_D\left(\tilde{V}, \tilde{Z}_D\right) &= g'\left(\frac{1}{2\tau} \left[\left(1 + \frac{4\tau^2}{\sigma_e^2\sigma_D^2}\right) \left(\frac{2\tau}{\sigma_e^2}\tilde{V} + \tilde{Z}_D \right) + \frac{1}{2\tau} \left(\int_0^1 D_{Si}^2 di + \tilde{Z}_S^2 \right) \right] \right).
\end{aligned}$$

We first show there is a unique equilibrium in the stock market for a fixed P_D. Note that we have:

$$P_S = \alpha_0 + \alpha_\mu \tilde{\mu} + \alpha_z \tilde{Z}_S.$$

This enables investors to invert the signal $\frac{P_S - \alpha_0}{\alpha_\mu} = \tilde{\mu} + \frac{\alpha_z}{\alpha_\mu}\tilde{Z}_S$ with precision $\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2} \sigma_S^{-2}$. Hence, applying the law of large numbers yields:

$$\begin{aligned} P_S &= \int_0^1 E(\tilde{x}|\Phi_i)\,di + \tau^{-1}\tilde{Z}_S\left(P_D + Var(\tilde{\mu}|\Phi_i)\right) \\ &= \int_0^1 \frac{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(\frac{P_S-\alpha_0}{\alpha_\mu}\right)+\sigma_\varepsilon^{-2}\tilde{\varphi}_i+\sigma_\mu^{-2}m}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}di + \tau^{-1}\tilde{Z}_S\left(\frac{1}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}+P_D\right) \\ &= \frac{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(\frac{P_S-\alpha_0}{\alpha_\mu}\right)+\sigma_\varepsilon^{-2}\tilde{\mu}+\sigma_\mu^{-2}m}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}} + \tau^{-1}\tilde{Z}_S\left(\frac{1}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}+P_D\right). \end{aligned}$$

Upon simplifying, we find:

$$\begin{aligned} P_S &= \frac{-\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\frac{\alpha_0}{\alpha_\mu}+\sigma_\varepsilon^{-2}\tilde{\mu}+\sigma_\mu^{-2}m}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}} \\ &\quad +\tau^{-1}\tilde{Z}_S\left(\frac{1}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}+P_D\right)\frac{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}. \end{aligned}$$

We thus have the following equilibrium conditions in the stock market for a fixed P_D:

$$\begin{aligned} \alpha_0 &= \frac{-\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\frac{\alpha_0}{\alpha_\mu}+\sigma_\mu^{-2}m}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}} \\ \alpha_\mu &= \frac{\sigma_\varepsilon^{-2}}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}} \\ \alpha_z &= \tau^{-1}P_D\frac{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}} + \frac{\tau^{-1}}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}. \end{aligned}$$

We proceed to prove that there is a unique vector $(\alpha_0, \alpha_\mu, \alpha_z)$ satisfying these three equations. Dividing the third equation by the second, we find:

$$\begin{aligned} \frac{\alpha_z}{\alpha_\mu} &= \frac{\tau^{-1}P_D\frac{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}} + \frac{\tau^{-1}}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}}{\frac{\sigma_\varepsilon^{-2}}{\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}} \\ &= \frac{\tau^{-1}P_D\left(\left(\frac{\alpha_z}{\alpha_\mu}\right)^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}\right)+\tau^{-1}}{\sigma_\varepsilon^{-2}}. \end{aligned}$$

Let $h \equiv \frac{\alpha_z}{\alpha_\mu}$. We prove there exists a unique, finite h that solves the above equation. The condition simplifies to:

$$F_1(h) \equiv -\tau h^3 \sigma_S^2 \sigma_\mu^2 + h^2 \sigma_S^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D h^2 \sigma_S^2 \sigma_\mu^2 + P_D h^2 \sigma_S^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2 = 0$$

Clearly, there are no negative solutions as $F_1'(x) < 0$ for $x < 0$ and $F_1(0) > 0$. By Descarte's rule of signs, there is at most one positive solution to this equation. Note that a positive solution exists because the left hand side approaches $-\infty$ as h goes to ∞. Finally, note that α_μ and α_z are uniquely defined by equilibrium conditions once h has been determined. To see this, note that the condition for α_μ may be written:[20]

$$\begin{aligned}\alpha_\mu &= \frac{\sigma_\varepsilon^{-2}}{h^{-2}\sigma_S^{-2}\left(1-\frac{1}{\alpha_\mu}\right)+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}} \\ &\Leftrightarrow \quad \alpha_\mu = \frac{\sigma_\mu^2\sigma_\varepsilon^2 + h^2\sigma_S^2\sigma_\mu^2}{\sigma_\mu^2\sigma_\varepsilon^2 + h^2\sigma_S^2\sigma_\mu^2 + h^2\sigma_S^2\sigma_\varepsilon^2}.\end{aligned}$$

It is easily seen that α_0 is also defined uniquely given α_μ and α_z. Next, consider the derivative price. Simplifying, we find:

$$\begin{aligned}P_D &= g'\left(\frac{1}{2\tau}\left[\left(1+\frac{4\tau^2}{\sigma_e^2\sigma_D^2}\right)\left(\frac{2\tau}{\sigma_e^2}\tilde{V}+\tilde{Z}_D\right)+\frac{1}{2\tau}\left(\int_0^1\left(\frac{E(\tilde{x}|\Phi_i)-\int_0^1 E(\tilde{x}|\Phi_i)\,di}{\tau^{-1}(Var(\mu|\Phi_i)+P_D)}-\tilde{Z}_S\right)^2 di+\tilde{Z}_S^2\right)\right]\right) \\ &= g'\left(\frac{1}{2\tau}\left[\left(1+\frac{4\tau^2}{\sigma_e^2\sigma_D^2}\right)\left(\frac{2\tau}{\sigma_e^2}\tilde{V}+\tilde{Z}_D\right)+\frac{1}{2\tau}\int_0^1\left(\frac{E(\tilde{x}|\Phi_i)-\int_0^1 E(\tilde{x}|\Phi_i)\,di}{\tau^{-1}(Var(\mu|\Phi_i)+P_D)}\right)^2 di\right]\right) \\ &= g'\left(\frac{1}{2\tau}\left[\left(1+\frac{4\tau^2}{\sigma_e^2\sigma_D^2}\right)\left(\frac{2\tau}{\sigma_e^2}\tilde{V}+\tilde{Z}_D\right)+\frac{h(P_D)^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2}{2\left(h(P_D)^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_D^2 h(P_D)\sigma_S^2\sigma_\mu^2+P_D^2 h(P_D)\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}\right]\right) \\ &= g'(\Gamma(P_D)).\end{aligned}$$

where $h(P_D)$ is defined as the implicit solution to $F_1(h(P_D)) = 0$ for any given P_D and $\Gamma(P_D) \equiv \frac{1}{2\tau}\left[\left(1+\frac{4\tau^2}{\sigma_e^2\sigma_D^2}\right)\left(\frac{2\tau}{\sigma_e^2}\tilde{V}+\tilde{Z}_D\right)+\frac{h(P_D)^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2}{2\left(h(P_D)^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_D^2 h(P_D)\sigma_S^2\sigma_\mu^2+P_D^2 h(P_D)\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}\right]$. Note that if there exists a unique solution P_D to this equation, then, by equilibrium condition (2), we can solve for the unique P_S. Hence, we will have proven that there exists a unique rational expectations equilibrium. To show that this is the case, we argue that:

$$\begin{aligned}\lim_{P_D\to-\infty} P_D - g'(\Gamma(P_D)) &= -\infty \\ \lim_{P_D\to\infty} P_D - g'(\Gamma(P_D)) &= \infty.\end{aligned}$$

[20]The zero solution to this equation may be ruled out by the fact that $\alpha_\mu = 0 \implies h = \infty$.

This holds because g' is continuous, and $\Gamma(P_D)$ is bounded in P_D. To see this, note that $h(P_D)$ is the unique solution to a cubic equation and is hence continuous. Moreover, it appears in both the numerator and denominator of $\frac{h(P_D)^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2}{2\left(h(P_D)^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_D^2h(P_D)\sigma_S^2\sigma_\mu^2+P_D^2h(P_D)\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}$ with the same power, while P_D appears in the denominator of $\frac{h(P_D)^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2}{2\left(h(P_D)^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_D^2h(P_D)\sigma_S^2\sigma_\mu^2+P_D^2h(P_D)\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}$ only. Now, applying the chain rule and implicit function theorem, we find that:

$$\begin{aligned}
&\frac{d}{dP_D}\Gamma(h,P_D)=\frac{d}{dP_D}\frac{h^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2}{2\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_Dh^2\sigma_S^2\sigma_\mu^2+P_Dh^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}\\
=\;&\frac{\partial}{\partial P_D}\frac{h^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2}{2\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_Dh^2\sigma_S^2\sigma_\mu^2+P_Dh^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}\\
&-\left(\frac{\partial F_1}{\partial h}\right)^{-1}\left(\frac{\partial F_1}{\partial P_D}\right)\frac{\partial}{\partial h}\frac{h^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2}{2\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_Dh^2\sigma_S^2\sigma_\mu^2+P_Dh^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}\\
=\;&\frac{-h^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2\left(h^2\sigma_S^2\sigma_\mu^2+h^2\sigma_S^2\sigma_\varepsilon^2+\sigma_\mu^2\sigma_\varepsilon^2\right)}{\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_Dh^2\sigma_S^2\sigma_\mu^2+P_Dh^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^3}\\
&-\frac{h^2\sigma_S^2\sigma_\mu^2+h^2\sigma_S^2\sigma_\varepsilon^2+\sigma_\mu^2\sigma_\varepsilon^2}{h\sigma_S^2\left(2\sigma_\mu^2\sigma_\varepsilon^2+2\sigma_\mu^2P_D+2\sigma_\varepsilon^2P_D+3h\tau\sigma_\mu^2\right)}\frac{2h^3\tau\sigma_S^4\sigma_\mu^6\sigma_\varepsilon^4P_D}{\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_Dh^2\sigma_S^2\sigma_\mu^2+P_Dh^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^3}\\
=\;&\frac{-h^2\tau\sigma_S^2\sigma_\mu^4\sigma_\varepsilon^2\left(h^2\sigma_S^2\sigma_\mu^2+h^2\sigma_S^2\sigma_\varepsilon^2+\sigma_\mu^2\sigma_\varepsilon^2\right)\left(3\tau h^3\sigma_S^2\sigma_\mu^2+2h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+2P_Dh^2\sigma_S^2\sigma_\mu^2+2P_Dh^2\sigma_S^2\sigma_\varepsilon^2+2P_D\sigma_\mu^2\sigma_\varepsilon^2\right)}{\left(2\sigma_\mu^2\sigma_\varepsilon^2+2\sigma_\mu^2P_D+2\sigma_\varepsilon^2P_D+3h\tau\sigma_\mu^2\right)\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_Dh^2\sigma_S^2\sigma_\mu^2+P_Dh^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^3}<0.
\end{aligned}$$

Thus, applying the fact that $g''>0$,[21]

$$\begin{aligned}
&\frac{\partial}{\partial P_D}\left(P_D-g'\left(\Gamma\left(P_D\right)\right)\right)\\
=\;&1-2g''\left(\Gamma\left(P_D\right)\right)\frac{d}{dP_D}\Gamma\left(P_D\right)>0.
\end{aligned}$$

■

Proof of Proposition (4). We have that:

$$\begin{aligned}
Vol_S&=\int_0^1\left|D_{Si}-\int_0^1 D_{Si}di\right|di\\
&=\int_0^1\left|\frac{E\left(\tilde{x}|\Phi_i\right)-\int_0^1E\left(\tilde{x}|\Phi_i\right)di-\tau^{-1}\tilde{Z}_S\left(P_D+Var\left(\tilde{\mu}|\Phi_i\right)\right)}{\tau^{-1}\left(P_D+Var\left(\tilde{\mu}|\Phi_i\right)\right)}+\tilde{Z}_S\right|di\\
&=\tau\frac{\int_0^1\left|E\left(\tilde{x}|\Phi_i\right)-\int_0^1E\left(\tilde{x}|\Phi_i\right)di\right|di}{P_D+Var\left(\tilde{\mu}|\Phi_i\right)}
\end{aligned}$$

[21]For a proof, see Lemma A7 in Breon-Drish (2015a).

and:

$$
\begin{aligned}
Vol_D &= \int_0^1 \left| \tau \left(\frac{1}{\sigma_e^2} \tilde{\eta}_i + \frac{2\tau}{\sigma_e^2 \sigma_D^2} \tilde{l} \right) - \tau g'^{-1}(P_D) + \frac{1}{2\tau} D_{Si}^2 - \int_0^1 D_{Di} di \right| di \\
&= \int_0^1 \left| \tau \left(\frac{1}{\sigma_e^2} \tilde{\eta}_i + \frac{2\tau}{\sigma_e^2 \sigma_D^2} \tilde{l} \right) - \tau g'^{-1}(P_D) + \frac{1}{2\tau} D_{Si}^2 - \int_0^1 \tau \left(\frac{1}{\sigma_e^2} \tilde{\eta}_i + \frac{2\tau}{\sigma_e^2 \sigma_D^2} \tilde{l} \right) - \tau g'^{-1}(P_D) + \frac{1}{2\tau} D_{Si}^2 \right| di \\
&= \int_0^1 \left| \tau \left(\frac{1}{\sigma_e^2} \tilde{\eta}_i - \tilde{V} \right) + \frac{1}{2\tau} D_{Si}^2 - \frac{1}{2\tau} \int_0^1 D_{Si}^2 di \right| di.
\end{aligned}
$$

■

Proof of Corollary (3). In order to take comparative statics, note that the equilibrium condition for P_D effectively involves the following two equations:

$$
\begin{aligned}
F_1 &\equiv -\tau h^3 \sigma_S^2 \sigma_\mu^2 + h^2 \sigma_S^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D h^2 \sigma_S^2 \sigma_\mu^2 + P_D h^2 \sigma_S^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2 = 0 \\
F_2 &\equiv P_D - g' \left(\frac{1}{2\tau} \left[\left(1 + \frac{4\tau^2}{\sigma_e^2 \sigma_D^2} \right) \left(\frac{2\tau}{\sigma_e^2} \tilde{V} + \tilde{Z}_D \right) + \frac{h^4 \tau \sigma_S^4 \sigma_\mu^4 \sigma_\varepsilon^2}{2 \left(h^2 \sigma_S^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D^2 h \sigma_S^2 \sigma_\mu^2 + P_D^2 h \sigma_S^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2 \right)^2} \right] \right).
\end{aligned}
$$

Applying the multivariate implicit function theorem, we find that for an arbitrary parameter γ, we have:

$$
\begin{aligned}
\begin{pmatrix} \frac{dh}{d\gamma} \\ \frac{dP_D}{d\gamma} \end{pmatrix} &= - \begin{pmatrix} \frac{\partial F_1}{\partial h} & \frac{\partial F_1}{\partial P_D} \\ \frac{\partial F_2}{\partial h} & \frac{\partial F_2}{\partial P_D} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial \gamma} \\ \frac{\partial F_2}{\partial \gamma} \end{pmatrix} \\
&= \frac{1}{\frac{\partial F_1}{\partial P_D} \frac{\partial F_2}{\partial h} - \frac{\partial F_1}{\partial h} \frac{\partial F_2}{\partial P_D}} \begin{pmatrix} \frac{\partial F_2}{\partial P_D} & -\frac{\partial F_1}{\partial P_D} \\ -\frac{\partial F_2}{\partial h} & \frac{\partial F_1}{\partial h} \end{pmatrix} \begin{pmatrix} \frac{\partial F_1}{\partial \gamma} \\ \frac{\partial F_2}{\partial \gamma} \end{pmatrix} \\
&= \frac{1}{\frac{\partial F_1}{\partial P_D} \frac{\partial F_2}{\partial h} - \frac{\partial F_1}{\partial h} \frac{\partial F_2}{\partial P_D}} \begin{pmatrix} \frac{\partial F_2}{\partial P_D} \frac{\partial F_1}{\partial \gamma} - \frac{\partial F_1}{\partial P_D} \frac{\partial F_2}{\partial \gamma} \\ -\frac{\partial F_2}{\partial h} \frac{\partial F_1}{\partial \gamma} + \frac{\partial F_1}{\partial h} \frac{\partial F_2}{\partial \gamma} \end{pmatrix}.
\end{aligned}
$$

Now, $\frac{1}{\frac{\partial F_1}{\partial P_D} \frac{\partial F_2}{\partial h} - \frac{\partial F_1}{\partial h} \frac{\partial F_2}{\partial P_D}} > 0$. To see this, note that:

$$
\begin{aligned}
& \frac{\partial F_1}{\partial P_D} \frac{\partial F_2}{\partial h} - \frac{\partial F_1}{\partial h} \frac{\partial F_2}{\partial P_D} \\
= \quad & \left(h^2 \sigma_S^2 \sigma_\mu^2 + h^2 \sigma_S^2 \sigma_\varepsilon^2 + \sigma_\mu^2 \sigma_\varepsilon^2 \right) \left(- \frac{h^3 \sigma_S^4 \sigma_\mu^6 \sigma_\varepsilon^4 P_D}{\left(h^2 \sigma_S^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D h^2 \sigma_S^2 \sigma_\mu^2 + P_D h^2 \sigma_S^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2 \right)^3} g'' \right) \\
& - \left(h \sigma_S^2 \left(2\sigma_\mu^2 \sigma_\varepsilon^2 + 2\sigma_\mu^2 P_D + 2\sigma_\varepsilon^2 P_D - 3h\tau \sigma_\mu^2 \right) \right) \left(1 + \frac{1}{2} \frac{h^4 \sigma_S^4 \sigma_\mu^4 \sigma_\varepsilon^2 \left(h^2 \sigma_S^2 \sigma_\mu^2 + h^2 \sigma_S^2 \sigma_\varepsilon^2 + \sigma_\mu^2 \sigma_\varepsilon^2 \right)}{\left(h^2 \sigma_S^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D h^2 \sigma_S^2 \sigma_\mu^2 + P_D h^2 \sigma_S^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2 \right)^3} g'' \right) \\
= \quad & -h \sigma_S^2 \left(2\sigma_\mu^2 \sigma_\varepsilon^2 + 2\sigma_\mu^2 P_D + 2\sigma_\varepsilon^2 P_D - 3h\tau \sigma_\mu^2 \right) + \frac{1}{2} \frac{\tau h^6 \sigma_S^6 \sigma_\mu^6 \sigma_\varepsilon^2 \left(h^2 \sigma_S^2 \sigma_\mu^2 + h^2 \sigma_S^2 \sigma_\varepsilon^2 + \sigma_\mu^2 \sigma_\varepsilon^2 \right)}{\left(h^2 \sigma_S^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D h^2 \sigma_S^2 \sigma_\mu^2 + P_D h^2 \sigma_S^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2 \right)^3} g''
\end{aligned}
$$

where the last line follows by substituting the equilibrium condition $F_1 = 0$. At an equilibrium h, we have

$h\sigma_S^2\left(2\sigma_\mu^2\sigma_\varepsilon^2+2\sigma_\mu^2 P_D+2\sigma_\varepsilon^2 P_D-3h\tau\sigma_\mu^2\right)=\frac{dF_1}{dh}<0$. This follows because $F_1(h)$ has a unique positive zero, and is positive when $h=0$. This, along with the fact that $g''>0$ proves that the expression is positive.

We next prove part ii) of the corollary. Note that:

$$\begin{aligned}\frac{dVol_D}{dV} &= \frac{dVol_D}{dP_D}\frac{dP_D}{dV}\\ \frac{dVol_D}{dZ_D} &= \frac{dVol_D}{dP_D}\frac{dP_D}{dZ_D}.\end{aligned}$$

By applying $\frac{dP_D}{dV}\propto-\frac{\partial F_2}{\partial h}\frac{\partial F_1}{\partial V}+\frac{\partial F_1}{\partial h}\frac{\partial F_2}{\partial V}$ and $\frac{dP_D}{dZ_D}\propto-\frac{\partial F_2}{\partial h}\frac{\partial F_1}{\partial V}+\frac{\partial F_1}{\partial h}\frac{\partial F_2}{\partial V}$, we can show that $\frac{dP_D}{dV}$ and $\frac{dP_D}{dZ_D}$ are positive. Therefore, the proof reduces to showing that $\frac{dVol_D}{dP_D}$ is positive. Now, Vol_D can be written as:

$$\begin{aligned}&\int_0^1\left|\frac{1}{\sigma_e^2}\left(\tilde{\eta}_i-\tilde{V}\right)+\frac{1}{2\tau^2}\left[D_{Si}^2-\int_0^1 D_{Si}^2 di\right]\right|di\\ =&\int_0^1\left|\frac{1}{\sigma_e^2}\tilde{e}_i+\frac{1}{2\tau^2}\left[D_{Si}^2-\int_0^1 D_{Si}^2 di\right]\right|di.\end{aligned}$$

Simplifying the second term in the absolute value, we find:

$$\begin{aligned}&D_{Si}^2-\int_0^1 D_{Si}^2 di\\ =&\left(\frac{\frac{\sigma_\varepsilon^{-2}\tilde{\varepsilon}_i}{h^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}}{\tau^{-1}\left(P_D+\left(h^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}\right)^{-1}\right)}-\tilde{Z}_S\right)^2\\ &-\int_0^1\left(\frac{h^2\tau\sigma_S^2\tilde{\varepsilon}_i\sigma_\mu^2}{h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_D h^2\sigma_S^2\sigma_\mu^2+P_D h^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2}-\tilde{Z}_S\right)^2 di\\ =&\left(\frac{\frac{\sigma_\varepsilon^{-2}\tilde{\varepsilon}_i}{h^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}}}{\tau^{-1}\left(P_D+\left(h^{-2}\sigma_S^{-2}+\sigma_\mu^{-2}+\sigma_\varepsilon^{-2}\right)^{-1}\right)}\right)^2-\left(\frac{h^2\tau\sigma_S^2\sigma_\mu^2}{h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_D h^2\sigma_S^2\sigma_\mu^2+P_D h^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2}\right)^2\sigma_\varepsilon^2\\ =&\frac{h^4\tau^2\sigma_S^4\sigma_\mu^4\left(\tilde{\varepsilon}_i^2-\sigma_\varepsilon^2\right)}{\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_D h^2\sigma_S^2\sigma_\mu^2+P_D h^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}\end{aligned}$$

and:

$$\begin{aligned}&\int_0^1\left|\frac{1}{\sigma_e^2}\tilde{e}_i+\frac{\tau^{-2}}{2}\frac{h^4\tau^2\sigma_S^4\sigma_\mu^4}{\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2+P_D h^2\sigma_S^2\sigma_\mu^2+P_D h^2\sigma_S^2\sigma_\varepsilon^2+P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2}\left(\tilde{\varepsilon}_i^2-\sigma_\varepsilon^2\right)\right|di\\ \equiv&\int_0^1\left|\frac{1}{\sigma_e^2}\tilde{e}_i+\aleph\left(\tilde{\varepsilon}_i^2-\sigma_\varepsilon^2\right)\right|di.\end{aligned}$$

Let $\hat{\varepsilon}_i \equiv \left(\tilde{\varepsilon}_i^2 - \sigma_\varepsilon^2\right)$. It can be checked that $\frac{d\aleph}{dP_D} < 0$. Hence, by the chain rule, we have that:

$$
\begin{aligned}
\frac{dVol_D}{dP_D} &= \frac{d}{dP_D}\int_0^1 \left|\frac{1}{\sigma_e^2}\tilde{e}_i + \aleph\left(\tilde{\varepsilon}_i^2 - \sigma_\varepsilon^2\right)\right| di \\
&\propto -\frac{\partial}{\partial\aleph}\int_0^1 \left|\frac{1}{\sigma_e^2}\tilde{e}_i + \aleph\left(\tilde{\varepsilon}_i^2 - \sigma_\varepsilon^2\right)\right| di \\
&= -\frac{\partial}{\partial\aleph}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\tilde{e}_i}{\sigma_e^2} + \aleph\hat{\varepsilon}_i\right| dF_{\hat{\varepsilon}_i} dF_{e_i} \\
&= -\frac{\partial}{\partial\aleph}\int_{-\infty}^{\infty}\left[\int_{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}^{\infty}\left(\frac{\tilde{e}_i}{\sigma_e^2} + \aleph\hat{\varepsilon}_i\right) dF_{\hat{\varepsilon}_i} - \int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\left(\frac{\tilde{e}_i}{\sigma_e^2} + \aleph\hat{\varepsilon}_i\right) dF_{\hat{\varepsilon}_i}\right] dF_{e_i} \\
&= -\int_{-\infty}^{\infty}\left[\int_{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}^{\infty}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} - \int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i}\right] dF_{e_i}.
\end{aligned}
$$

Note that:

$$
\begin{aligned}
\int_{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}^{\infty}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} + \int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} &= 0 \\
&\implies \int_{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}^{\infty}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} = -\int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} \\
&\implies \int_{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}^{\infty}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} - \int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} = -2\int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i}.
\end{aligned}
$$

We thus have:

$$
\begin{aligned}
& -\int_{-\infty}^{\infty}\left[\int_{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}^{\infty}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} - \int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i}\right] dF_{e_i} \\
= \; & 2\int_{-\infty}^{\infty}\int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} dF_{e_i} < 0
\end{aligned}
$$

since $\int_{-\infty}^{-\frac{\tilde{e}_i}{\sigma_e^2\aleph}}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} < \int_{-\infty}^{\infty}\hat{\varepsilon}_i dF_{\hat{\varepsilon}_i} = E\left[\hat{\varepsilon}_i\right] = 0$. This completes the proof. ■

Proof of Corollary (4). First, note that:

$$
\begin{aligned}
Vol_S &= \tau\frac{\int_0^1\left|E\left(\tilde{\mu}|\Phi_i\right) - \int_0^1 E\left(\tilde{\mu}|\Phi_i\right) di\right| di}{P_D + Var\left(\tilde{\mu}|\Phi_i\right)} = \tau\frac{\int_0^1\left|\frac{\sigma_i^{-2}\left(\tilde{\varphi}_i - \tilde{\mu}\right)}{h^{-2}\sigma_S^{-2} + \sigma_i^{-2} + \sigma_\mu^{-2}}\right|}{P_D + \left(h^{-2}\sigma_S^{-2} + \sigma_i^{-2} + \sigma_\mu^{-2}\right)^{-1}} \\
&= \frac{2}{\sqrt{\pi}}\tau\frac{\frac{\sigma_i^{-1}}{h^{-2}\sigma_S^{-2} + \sigma_i^{-2} + \sigma_\mu^{-2}}}{P_D + \left(h^{-2}\sigma_S^{-2} + \sigma_i^{-2} + \sigma_\mu^{-2}\right)^{-1}} \\
&= \frac{2}{\sqrt{\pi}}\tau\frac{h^2\sigma_S^2\sigma_i\sigma_\mu^2}{h^2\sigma_S^2\sigma_i^2\sigma_\mu^2 + P_D h^2\sigma_S^2\sigma_i^2 + P_D h^2\sigma_S^2\sigma_\mu^2 + P_D\sigma_i^2\sigma_\mu^2}.
\end{aligned}
$$

In order to complete the proof, we consider the directional effect of a change in each of the random parameters of the model, $\tilde{Z}_S$, $\tilde{Z}_D$, $\tilde{V}$, and $\tilde{\mu}$, on ΔP_S, ΔP_D, Vol_S, and Vol_D. First, consider $\tilde{Z}_S$. We have that

$$\begin{aligned}
\frac{dVol_S}{dZ_S} &= 0;\ \frac{dVol_D}{dZ_S} = 0 \\
\frac{dP_S}{dZ_S} &> 0;\ \frac{dP_D}{dZ_S} = 0.
\end{aligned}$$

Next, consider $\tilde{Z}_D$. One can check that:

$$\begin{aligned}
\frac{dVol_S}{dZ_D} &= \frac{dVol_S}{dP_D}\frac{dP_D}{dZ_D} < 0 \\
\frac{dVol_D}{dZ_D} &= \frac{dVol_D}{dP_D}\frac{dP_D}{dZ_D} < 0 \\
\frac{dP_S}{dZ_D} &< 0;\ \frac{dP_D}{dZ_D} > 0.
\end{aligned}$$

Similarly, we have that:

$$\begin{aligned}
\frac{dVol_S}{dV} &= \frac{dVol_S}{dP_D}\frac{dP_D}{dV} < 0 \\
\frac{dVol_D}{dV} &= \frac{dVol_D}{dP_D}\frac{dP_D}{dV} < 0 \\
\frac{dP_S}{dV} &< 0;\ \frac{dP_D}{dV} > 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{dVol_S}{d\mu} &= 0;\ \frac{dVol_D}{d\mu} = 0 \\
\frac{dP_S}{d\mu} &> 0;\ \frac{dP_D}{d\mu} = 0.
\end{aligned}$$

Combining these forces, we see that any change in the underlying random variables that causes the stock price to increase causes volume in both markets to either remain constant or increase. Likewise, any change in an underlying random variable that causes the derivative price to increase leads volume in both markets to remain constant or decrease. This completes the proof. ■

Proof of Proposition (5). Let $\sigma_m^2 \equiv \int_0^1 \left(m_i - \int_0^1 m_i di\right)^2 di$. It is straightforward to replicate the above proofs in the case of heterogenous prior beliefs to show that a unique equilibrium exists, which satisfies the following two equations:

$$
\begin{aligned}
0 &= F_1 = -\tau h^3\sigma_S^2\sigma_\mu^2 + h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2 + P_D h^2\sigma_S^2\sigma_\mu^2 + P_D h^2\sigma_S^2\sigma_\varepsilon^2 + P_D\sigma_\mu^2\sigma_\varepsilon^2 \\
0 &= F_2^{\sigma_m^2} \equiv P_D - \\
&\quad g'\left(2\tau\left(\frac{1}{\sigma_e^2}\tilde{V} + \frac{1}{\sigma_D^2}\tilde{l}\right) + \tilde{Z}_D\right. \\
&\quad \left. + \frac{1}{2\tau}\left(\frac{h^4\tau\sigma_S^4\sigma_\mu^4\sigma_\varepsilon^2}{2\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2 + P_D h^2\sigma_S^2\sigma_\mu^2 + P_D h^2\sigma_S^2\sigma_\varepsilon^2 + P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^2} + \frac{\sigma_\mu^{-2}\sigma_m^2}{\tau^{-2}\left(\left(h^{-2}\sigma_S^{-2} + \sigma_\mu^{-2} + \sigma_\varepsilon^{-2}\right)^{-1} + P_D\right)}\right)\right).
\end{aligned}
$$

Applying the implicit function theorem as in the proof of Corollary 3, we have that:

$$
\begin{aligned}
\frac{\partial P_D}{\partial \sigma_m^2} &\propto -\frac{\partial F_2^{\sigma_m^2}}{\partial h}\frac{\partial F_1}{\partial \sigma_m^2} + \frac{\partial F_1}{\partial h}\frac{\partial F_2^{\sigma_m^2}}{\partial \sigma_m^2} \\
&= \frac{\partial F_1}{\partial h}\frac{\partial F_2^{\sigma_m^2}}{\partial \sigma_m^2} \propto -\frac{\partial F_2^{\sigma_m^2}}{\partial \sigma_m^2} > 0
\end{aligned}
$$

since $\frac{\partial F_1}{\partial \sigma_m^2} = 0$ and $\frac{\partial F_1}{\partial h} < 0$. To see the effect of σ_m^2 on the expected stock price, note that:

$$
\frac{dE\left(P_S\right)}{d\sigma_m^2} \propto -\tau^{-1}\bar{z}_S\frac{dE\left(P_D\right)}{d\sigma_m^2}.
$$

Since, for any realization of P_D, $\frac{dE(P_D)}{d\sigma_m^2} > 0$, we have that this is negative. ■

Proof of Corollary (5). Recall from the proof of Corollary 3, $\frac{dP_D}{d\sigma_\varepsilon^2} \propto -\frac{\partial F_2}{\partial h}\frac{\partial F_1}{\partial \sigma_\varepsilon^2} + \frac{\partial F_1}{\partial h}\frac{\partial F_2}{\partial \sigma_\varepsilon^2}$. Simplifying, we find:

$$
\begin{aligned}
\frac{dP_D}{d\sigma_\varepsilon^2} &\propto -\frac{2h^3\tau\sigma_S^4\sigma_\mu^6\sigma_\varepsilon^4 P_D\left(h^2\sigma_S^2\sigma_\mu^2 + P_D h^2\sigma_S^2 + P_D\sigma_\mu^2\right)}{\left(h^2\sigma_S^2\sigma_\mu^2\sigma_\varepsilon^2 + P_D h^2\sigma_S^2\sigma_\mu^2 + P_D h^2\sigma_S^2\sigma_\varepsilon^2 + P_D\sigma_\mu^2\sigma_\varepsilon^2\right)^3} + g'' * \\
&\quad \frac{h^4\tau\sigma_S^4\sigma_\mu^4 h\sigma_S^2\left(2\sigma_\mu^2\sigma_\varepsilon^2 + 2\sigma_\mu^2 P_D + 2\sigma_\varepsilon^2 P_D - 3h\tau\sigma_\mu^2\right)\left(\sigma_\varepsilon^2\sigma_\mu^2 P_D - h^2\sigma_S^2\sigma_\mu^2 P_D + h^2\sigma_\varepsilon^2\sigma_S^2 P_D + h^2\sigma_\varepsilon^2\sigma_S^2\sigma_\mu^2\right)}{2\left(\sigma_\varepsilon^2\sigma_\mu^2 P_D + h^2\sigma_S^2\sigma_\mu^2 P_D + h^2\sigma_\varepsilon^2\sigma_S^2 P_D + h^2\sigma_\varepsilon^2\sigma_S^2\sigma_\mu^2\right)^3}
\end{aligned}
$$

which has sign that is an ambiguous function of the underlying parameters. Next, we have that $\frac{dP_D}{d\sigma_\mu^2} \propto -\frac{\partial F_2}{\partial h}\frac{\partial F_1}{\partial \sigma_\mu^2} + \frac{\partial F_1}{\partial h}\frac{\partial F_2}{\partial \sigma_\mu^2}$. Note that from the proof of Corollary 3, $\frac{\partial F_1}{\partial h} < 0$ and $\frac{\partial F_2}{\partial h} > 0$. Moreover, we have that:

$$
\begin{aligned}
\frac{\partial F_1}{\partial \sigma_\mu^2} &= -\tau h^3\sigma_S^2 + h^2\sigma_S^2\sigma_\varepsilon^2 + P_D h^2\sigma_S^2 + P_D\sigma_\varepsilon^2 \\
&= -h^2\frac{\sigma_S^2}{\sigma_\mu^2}\sigma_\varepsilon^2 P_D < 0
\end{aligned}
$$

where we substituted the equilibrium condition $F_1 = 0$. Moreover,

$$\begin{aligned}
\frac{\partial F_2}{\partial \sigma_\mu^2} &\propto -\frac{\partial}{\partial \sigma_\mu^2} \frac{h^4 \tau^2 \sigma_S^4 \sigma_\mu^4 \sigma_\varepsilon^2}{\left(h^2 \sigma_S^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D h^2 \sigma_S^2 \sigma_\mu^2 + P_D h^2 \sigma_S^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2\right)^2} \\
&= -\frac{2\tau^2 h^6 \sigma_\mu^2 \sigma_S^6 \sigma_\varepsilon^4 P_D}{\left(\sigma_\mu^2 \sigma_\varepsilon^2 P_D + h^2 \sigma_S^2 \sigma_\varepsilon^2 P_D + h^2 \sigma_\mu^2 \sigma_S^2 P_D + h^2 \sigma_\mu^2 \sigma_S^2 \sigma_\varepsilon^2\right)^3} < 0,
\end{aligned}$$

proving that $\frac{dP_D}{d\sigma_\mu^2} > 0$. Finally, we have that $\frac{\partial P_D}{\partial \sigma_S^2} \propto -\frac{\partial F_2}{\partial h}\frac{\partial F_1}{\partial \sigma_\mu^2} + \frac{\partial F_1}{\partial h}\frac{\partial F_2}{\partial \sigma_\mu^2}$. Notice that

$$\frac{\partial F_1}{\partial \sigma_S^2} = -\tau h^3 \sigma_\mu^2 + h^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D h^2 \sigma_\mu^2 + P_D h^2 \sigma_\varepsilon^2 = -\frac{P_D \sigma_\mu^2 \sigma_\varepsilon^2}{\sigma_S^2} < 0$$

where we substituted $F_1 = 0$. Next,

$$\begin{aligned}
\frac{\partial F_2}{\partial \sigma_S^2} &\propto -\frac{\partial}{\partial \sigma_S^2} \frac{h^4 \tau^2 \sigma_S^4 \sigma_\mu^4 \sigma_\varepsilon^2}{\left(h^2 \sigma_S^2 \sigma_\mu^2 \sigma_\varepsilon^2 + P_D h^2 \sigma_S^2 \sigma_\mu^2 + P_D h^2 \sigma_S^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2\right)^2} \\
&= -\frac{-2h^4 \sigma_S^2 \tau^2 \sigma_\mu^6 \sigma_\varepsilon^4 P_D}{\left(\sigma_S^2 h^2 \sigma_\mu^2 \sigma_\varepsilon^2 + \sigma_S^2 P_D h^2 \sigma_\mu^2 + \sigma_S^2 P_D h^2 \sigma_\varepsilon^2 + P_D \sigma_\mu^2 \sigma_\varepsilon^2\right)^3} < 0,
\end{aligned}$$

proving that $\frac{\partial P_D}{\partial \sigma_S^2} > 0$.

ii) The first result follows from the numerical example in figure 2. The second result follows from applying the chain rule:

$$\frac{dE[P_S]}{d\sigma_S^2} \propto -\left(\frac{dE[Var(\tilde{\mu}|\Phi_i)]}{d\sigma_S^2} + E\left[\frac{dP_D}{d\sigma_S^2}\right]\right) \tag{15}$$

$$\frac{dE[P_S]}{d\sigma_\mu^2} \propto -\left(\frac{dE[Var(\tilde{\mu}|\Phi_i)]}{d\sigma_\mu^2} + E\left[\frac{dP_D}{d\sigma_\mu^2}\right]\right). \tag{16}$$

By the proof of part i), $E\left[\frac{dP_D}{d\sigma_S^2}\right] > 0$ and $E\left[\frac{dP_D}{d\sigma_\mu^2}\right] > 0$. ■

Proof of Corollary (6). This follows since:

$$\begin{aligned}
\frac{dVar(\tilde{\mu}|\Phi_i)}{dP_D} &= \frac{d}{dP_D}\frac{1}{h^{-2}\sigma_S^{-2} + \sigma_\mu^{-2} + \sigma_\varepsilon^{-2}} \\
&= \frac{2h\sigma_S^2 \sigma_\mu^4 \sigma_\varepsilon^4}{\left(h^2 \sigma_S^2 \sigma_\mu^2 + h^2 \sigma_S^2 \sigma_\varepsilon^2 + \sigma_\mu^2 \sigma_\varepsilon^2\right)^2}\frac{dh}{dP_D} \\
&= -\frac{2h\sigma_S^2 \sigma_\mu^4 \sigma_\varepsilon^4}{\left(h^2 \sigma_S^2 \sigma_\mu^2 + h^2 \sigma_S^2 \sigma_\varepsilon^2 + \sigma_\mu^2 \sigma_\varepsilon^2\right)^2}\left(\frac{dF_1}{dh}\right)^{-1}\frac{dF_1}{dP_D} \\
&= -\frac{2h\sigma_S^2 \sigma_\mu^4 \sigma_\varepsilon^4}{\left(h^2 \sigma_S^2 \sigma_\mu^2 + h^2 \sigma_S^2 \sigma_\varepsilon^2 + \sigma_\mu^2 \sigma_\varepsilon^2\right)^2}\frac{\tau h^2 \sigma_S^2 \sigma_\mu^2 + \tau h^2 \sigma_S^2 \sigma_\varepsilon^2 + \tau \sigma_\mu^2 \sigma_\varepsilon^2}{h\sigma_S^2\left(2\sigma_\mu^2 \sigma_\varepsilon^2 + 2\tau \sigma_\mu^2 P_D + 2\tau \sigma_\varepsilon^2 P_D + 3h\tau \sigma_\mu^2\right)} > 0.
\end{aligned}$$

■

Proof of Proposition (6). i) This follows from the proofs of Proposition 5 and Corollary 5.

ii) Letting $P_D = \int_0^1 E\left(\tilde{V}|\Phi_i\right) di + VRP$, the proof follows by replicating the proof of 3 holding fixed $\int_0^1 E\left(\tilde{V}|\Phi_i\right) di$.

iii) This follows from an examination of expression 2. ■

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