Dynamic Portfolio Execution – Detailed Proofs

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1 Proofs

Lemma 1 (Temporary Price Impact)

A buy order of size $x$ being executed against $i$’s ask-side inventory $q_i^a$, displaces the best ask price from $a_{i,n} → a_{i,n}^*$, according to:

$$\int_{a_{i,n}}^{a_{i,n}^*} q_i^a(p) dp = x.$$

Combining this expression with Assumption 1, we have

$$\int_{a_{i,n}}^{a_{i,n}^*} q_i^a 1_{p \geq a_{i,n}} dp = q_i^a (a_{i,n}^* - a_{i,n}) = x \Rightarrow a_{i,n}^*(x) = a_{i,n} + \frac{x}{q_i^a}.$$

Therefore for $x = x_{i,n}^+$, we have $a_{i,n}^* = a_{i,n} + \frac{x_{i,n}^+}{q_i}$ and the temporary price impact displacement is defined as $f_{i,n}^a = a_{i,n}^* - a_{i,n} = \frac{x_{i,n}^+}{q_i}$. Similarly, for a sell order of size $x$, we have

$$\int_{b_{i,n}^*}^{b_{i,n}} q_i^b 1_{p \leq b_{i,n}} dp = q_i^b (b_{i,n} - b_{i,n}^*) = x \Rightarrow b_{i,n}^*(x) = b_{i,n} - \frac{x}{q_i^b}.$$

Therefore for $x = x_{i,n}^-$, we have $b_{i,n}^* = b_{i,n} - \frac{x_{i,n}^-}{q_i}$ and $b_{i,n} - b_{i,n}^* = \frac{x_{i,n}^-}{q_i^b}$. ■

Lemma 2 (Best Prevailing Bid/Ask-Prices)

We present below the derivation for the dynamics of the best ask price; the dynamics for the best bid price are derived in a similar way. The exponential decay Assumption 4, specifies the dynamics of the best ask price over period $\tau$. The best ask price at $n$, before the trade at $n$ arrives, depends on the previous displacement

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from the trade at $n - 1$, $a_{i,n-1}^*$, and on how quickly new limit orders arrives and push the book towards the next steady state, $a_{i,n-}^\infty$. The notation $n^-$ defines the time immediately preceding both the arrival of the trade and the realization of the random walk at $n$, i.e. $a_{i,n}^\infty = a_{i,n-}^\infty + \epsilon_{i,n}$. More specifically, we have

$$a_{i,n} = a_{i,n}^\infty + (a_{i,n-1}^* - a_{i,n-}^\infty)e^{-\rho_i^a \tau}.$$  \hspace{1cm} (25)$$

Executing an order at $n - 1$, the temporary price impact denoted $f_{i,n-1}^a = \frac{x_{i,n}}{q_i}$ will move $i$’s best ask price to

$$a_{i,n-1}^* = a_{i,n-1} + f_{i,n-1}^a.$$ \hspace{1cm} (26)$$

The executed order also has a net permanent price impact denoted $g_{i,n-1} = \sum_{j \in I_M} \lambda_{ij}(x_{j,n-1}^+ - x_{j,n-1}^-)$ on $i$’s mid-price. Further, the new steady-state ask price at time $n$, before the trade arrives at $n$, is given by

$$a_{i,n-}^\infty = a_{i,n-1} + g_{i,n-1},$$ \hspace{1cm} (27)$$

Plugging the expressions from (26) and (27) into equation (25), we obtain

$$a_{i,n} = a_{i,n}^\infty + (a_{i,n-1}^* - a_{i,n-}^\infty)e^{-\rho_i^a \tau}$$

$$= a_{i,n}^\infty + (a_{i,n-1} + f_{i,n-1}^a - (a_{i,n-1}^\infty + g_{i,n-1}))e^{-\rho_i^a \tau}$$

$$= a_{i,n}^\infty + (a_{i,n-1} - a_{i,n-1}^\infty + f_{i,n-1}^a - g_{i,n-1})e^{-\rho_i^a \tau}.$$  \hspace{1cm} (28)$$

Then, defining the functions

$$d_{i,n-1}^a = a_{i,n-1} - a_{i,n-1}^\infty$$ \hspace{1cm} (29)$$

and

$$\kappa_i^a(x_{i,n-1}^+, x_{i,n-1}^-) = \kappa_i^a(x_{i,n-1}^\pm) = f_{i,n-1}^a - g_{i,n-1},$$ \hspace{1cm} (30)$$

the price $a_{i,n}$ can be written as

$$a_{i,n} = a_{i,n}^\infty + (d_{i,n-1}^a + \kappa_i^a(x_{i,n-1}^\pm))e^{-\rho_i^a \tau}$$ \hspace{1cm} (31)$$

From equation (30), we have $a_{i,n} - a_{i,n}^\infty = (d_{i,n-1}^a + \kappa_i^a(x_{i,n-1}^\pm))e^{-\rho_i^a \tau}$ and from equation (28), we have

$$a_{i,n} - a_{i,n}^\infty = d_{i,n}^a.$$ \hspace{1cm} (32)$$

Therefore, combining these two equations, we obtain

$$d_{i,n}^a = (d_{i,n-1}^a + \kappa_i^a(x_{i,n-1}^\pm))e^{-\rho_i^a \tau},$$ \hspace{1cm} (33)$$
which is the recursive form of the state variable \( d_{i,n}^a \) given in Assumption 4. Assuming all the order books are originally full, i.e., \( d_{i,0}^a = 0, \forall i \in I_M \), we equivalently have in non-recursive form

\[
d_{i,n}^a = \sum_{k=1}^{n} \kappa_i^a(x_{i,n-1}^\pm) e^{-\rho_i^a(n-k+1)\tau}.
\]

Note, \( \kappa_i^a(x_{i,n-1}^\pm) \) is given by

\[
\kappa_i^a(x_{i,n-1}^\pm) = f_{i,k}^a - g_{i,k} = \frac{x_{i,k-1}^+}{q_i^a} - \sum_{j \in I_M} \lambda_{ij} (x_{j,k-1}^+ - x_{j,k-1}^-) = \frac{x_{i,k-1}^+}{q_i^a} - \sum_{j \in I_M} \lambda_{ij} \delta x_{j,k-1}, \quad (33)
\]

where \( \delta x_{j,k-1} = x_{j,k-1}^+ - x_{j,k-1}^- \). Therefore, replacing \( \kappa_i^a(x_{i,n-1}^\pm) \) by its explicit form in equation (33), we have

\[
d_{i,n}^a = \sum_{k=1}^{n} \left( \frac{x_{i,k-1}^+}{q_i^a} - \sum_{j \in I_M} \lambda_{ij} \delta x_{j,k-1} \right) e^{-\rho_i^a(n-k+1)\tau}. \quad (34)
\]

Next, we proceed to calculate the best ask price at a general time \( n \). From equation (31), we have

\[
a_{i,n} = a_{i,n}^\infty + d_{i,n}^a, \quad \text{where the steady-state price } a_{i,n}^\infty \text{ is given in Assumption 3. Specifically, } a_{i,n}^\infty = v_{i,n} + \frac{1}{2} s_i \quad \text{where } v_{i,n} \text{ is defined in Assumption 5 and takes the form}
\]

\[
v_{i,n} = u_{i,n} + \sum_{j \in I_M} \lambda_{ij} \sum_{k=1}^{n} (x_{j,k-1}^+ - x_{j,k-1}^-).
\]

Therefore, we have

\[
a_{i,n} = a_{i,n}^\infty + d_{i,n}^a = u_{i,n} + \frac{1}{2} s_i + \sum_{j \in I_M} \lambda_{ij} \sum_{k=1}^{n} (x_{j,k-1}^+ - x_{j,k-1}^-) + d_{i,n}^a.
\]

Finally, combining the above with the expression in equation (34), and extending the analysis to \( m \) assets and to the bid side, we have

\[
a_{i,n} = u_{i,n} + \frac{1}{2} s_i + \sum_{k=1}^{n} \sum_{j \in I_M} \lambda_{ij} \delta x_{j,k-1} + \sum_{k=1}^{n} \left( \frac{x_{i,k-1}^+}{q_i^a} - \sum_{j \in I_M} \lambda_{ij} \delta x_{j,k-1} \right) e^{-\rho_i^a(n-k+1)\tau}, \quad (35)
\]

\[
b_{i,n} = u_{i,n} - \frac{1}{2} s_i + \sum_{k=1}^{n} \sum_{j \in I_M} \lambda_{ij} \delta x_{j,k-1} + \sum_{k=1}^{n} \left( -\frac{x_{i,k-1}^-}{q_i^a} - \sum_{j \in I_M} \lambda_{ij} \delta x_{j,k-1} \right) e^{-\rho_i^a(n-k+1)\tau}. \quad (36)
\]

Next, we proceed to rewrite these expressions in vector form, utilizing the recursive expressions. The state vectors \( \mathbf{z} \) and \( \mathbf{d} \) introduced in the text follow naturally by looking at the terms in the two previous equations (35) and (36). Letting \( \mathbf{z}_{i,n} \) be the state vector of net shares remaining to be purchased right before the next
order arrives at time $n$, we have by definition

$$z_{n} = z_{0} - \sum_{k=1}^{n}(x_{k}^{+} - x_{k}^{-}) = z_{0} - \sum_{k=1}^{n} \Delta M x_{k-1}.$$  

Recursively, we can write $z_{n} = z_{n-1} - \Delta M x_{n-1}$. Using this recursive form, the PPI term in the price process equations (35) and (36) can be written as

$$\sum_{k=1}^{n} \sum_{j \in I_M} \lambda_{ij} \delta x_{j,k-1} = [\Lambda(z_{0} - z_{n})]_{i},$$

where $[\cdot]_{i}$ is an operator which returns the $i$-th line of a matrix and $\Lambda$ is the matrix of PPI factors.

Using Assumption 4, the second state vector can be written in vector form as

$$d_{n}^{a} = (d_{n-1}^{a} + \kappa^{a} x_{n-1}) e^{-p^{a}},$$

Similarly for the bid side we have $d_{n}^{b} = (d_{n-1}^{b} + \kappa^{b} x_{n-1}) e^{-p^{b}}$. Combining the above results, and identifying the terms in equations (35) and (36), we have the final vector forms for the best available ask and bid prices

$$a_{n} = u_{n} + \frac{1}{2} s_{n} + \Lambda(z_{0} - z_{n}) + d_{n}^{a},$$

$$b_{n} = u_{n} - \frac{1}{2} s_{n} + \Lambda(z_{0} - z_{n}) + d_{n}^{b}.$$  

**Lemma 3 (Execution Costs/Revenues)**

Following an executed order, the associated costs/revenues can simply be calculated by integrating the best available bid/ask prices over the total amount of units executed $x$. It follows that

$$c_{i,n}(x) = \int_{0}^{x} a_{i,n}^{*}(u) du \text{ and } r_{i,n}(x) = \int_{0}^{x} b_{i,n}^{*}(u) du,$$

where $a_{i,n}^{*}(x)$ and $b_{i,n}^{*}(x)$ are given in Lemma 1. Therefore, we have

$$c_{i,n}(x) = \int_{0}^{x} a_{i,n}^{*}(u) du = \int_{0}^{x} (a_{i,n} + \frac{u}{q^{a}_{i}}) du = (a_{i,n} x + \frac{x^2}{2q^{a}_{i}}),$$

$$r_{i,n}(x) = \int_{0}^{x} b_{i,n}^{*}(u) du = \int_{0}^{x} (b_{i,n} - \frac{u}{q^{b}_{i}}) du = (b_{i,n} x - \frac{x^2}{2q^{b}_{i}}).$$
Specifically, for an incoming buy order \( x = x^+_{i,n} \) or sell order \( x = x^-_{i,n} \), we have costs and revenues given by

\[
c_i,n(x^+_{i,n}) = \left( a_i,n + \frac{x^+_{i,n}}{2q^a_i} \right) x^+_{i,n} \quad \text{and} \quad r_i,n(x^-_{i,n}) = \left( b_i,n - \frac{x^-_{i,n}}{2q^b_i} \right) x^-_{i,n}.
\]

The vector forms across all assets at time \( n \) can then be expressed as

\[
c_n = \begin{bmatrix} x^+_{n} \\ x^-_{n} \end{bmatrix} = \begin{bmatrix} a_{n} \\ b_{n} \end{bmatrix} + Q^a \begin{bmatrix} x^+_{n} \\ x^-_{n} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad r_n = \begin{bmatrix} x^+_{n} \\ x^-_{n} \end{bmatrix} = \begin{bmatrix} b_{n} - Q^b \begin{bmatrix} x^+_{n} \\ x^-_{n} \end{bmatrix} \end{bmatrix}.
\]

**Proposition 1 (Path Independence)**

We first prove that path independence holds for a simplified unconstrained version of the problem. From there, it becomes clear that this result can be extended to the original constrained problem. Key to the proof will be a separability property of the wealth and value functions at each period, which can be separated into a linear stochastic term and a deterministic function of the controls and state variables. The reward associated with the manager’s buy and sell orders at \( n \) is

\[
\pi_n = x_{n}^+ (b_{n} - Q^b x_{n}^-) - x_{n}^+ (a_{n} + Q^a x_{n}^+) = x_{n}^+ p_{n} - x_{n}^+ Q x_{n},
\]

where we introduce the following notation:

\[
x_{n} = \begin{bmatrix} x_{n}^+ \\ x_{n}^- \end{bmatrix}; \quad Q = \begin{bmatrix} Q^a & 0 \\ 0 & Q^b \end{bmatrix}; \quad p_{n} = \begin{bmatrix} -a_{n} \\ b_{n} \end{bmatrix} = \left[ -(u_{n} + \frac{1}{2}s_{n} + \Lambda(z_{0} - z_{n}) + d^a_{n}) \\ u_{n} - \frac{1}{2}s_{n} + \Lambda(z_{0} - z_{n}) + d^b_{n} \right].
\]

Isolating the terms which depend explicitly on the noise term \( u_{n} \), the reward can be written as

\[
\pi_n = \theta_n - x_{n}^+ \Delta M u_{n}, \quad (37)
\]

where the function \( \theta_n = \theta_n(x_{n}, d_{n}, x_{n}) \) is by construction, a deterministic function linear in the state vectors, and quadratic in the controls at \( n \). Formally, \( \theta_n \) represents the manager’s execution costs for an order at time \( n \), net of the exogenous stochastic parameters of the problem: \( \theta_n = x_{n}^+ p_{n} - x_{n}^+ Q x_{n} + x_{n}^+ \Delta M u_{n} \).

It will be useful for the proof to write \( \theta_n \) in a quadratic form given by \( \theta_n = x_{n}^+ \Psi \theta_n x_{n} + x_{n}^+ \psi \theta_n \), for appropriately chosen matrix \( \Psi \theta_n \) and vector \( \psi \theta_n \). The manager’s cumulative wealth at an arbitrary time \( n \) is \( W_n = \sum_{j=0}^{n} \pi_j \). Recursively, we have

\[
W_n = W_{n-1} + \pi_n. \quad (38)
\]

The manager’s optimization problem over his terminal wealth is

\[
J_0 = \max_x \mathbb{E}_0 [ -e^{-\alpha W_N} ],
\]
where $J_0(\cdot)$ denotes the value function at 0. We can gain some useful insights into the properties of the value function and optimal control by looking at the first two iterations, starting at $N$.

**Analysis at the boundary**

By definition, at the final time step, the optimal policy is equal to the state vector of remaining trades, i.e., $\mathbf{x}_{J,N}^* \Delta_M = \mathbf{z}_{j,N}^*$. Then, using the equations (37) and (38), the value function at $N$ is

$$J_N^* = -e^{-\alpha W_N} = -e^{-\alpha(W_{N-1} + \pi_N)} = -e^{-\alpha(W_{N-1} + \theta_N^* - \mathbf{x}_{J,N}^* \Delta_M u_{N,N})} = -e^{-\alpha(W_{N-1} + \theta_N^* - \mathbf{z}_{j,N}^* u_{N,N})},$$

(39)

where $\pi_N^* = \theta_N^* - \mathbf{x}_{J,N}^* \Delta_M u_{N,N}$ and $\theta_N^* = \theta_N(\mathbf{z}_{i,N}, \mathbf{d}_{i,N}, \mathbf{x}_{J,N}^*) = \theta_N^*(\mathbf{z}_{i,N}, \mathbf{d}_{i,N})$. Rolling one step back, we show that $J_{N-1}$ only depends on the known cumulative wealth from the previous period, $N-2$, and the state vectors of the problem at $N-1$. Given $\mathbf{x}_{J,N}$, the value function at $N-1$ is

$$J_{N-1} = \max_{\mathbf{x}_{J,N-1}} \mathbb{E}_{N-1} \left[ -e^{-\alpha W_N} \right].$$

From equation (38), we have $W_N = W_{N-2} + \pi_{N-1} + \pi_N^*$. Then, applying equation (37) we obtain

$$J_{N-1} = \max_{\mathbf{x}_{J,N-1}} \mathbb{E}_{N-1} \left[ -e^{-\alpha(W_{N-2} + \theta_{N-1} - \mathbf{x}_{J,N-1} \Delta_M u_{N-1}) + (\theta_N^* - \mathbf{z}_{j,N-1} \Delta_M u_{N-1})} \right].$$

(40)

At this stage, we can remove the dependence on the state at time $N$ by using the recursive state equations: We know that $\mathbf{x}_{J,N}^* \Delta_M = \mathbf{z}_{j,N}^* = \mathbf{z}_{j,N-1} - \mathbf{x}_{J,N-1} \Delta_M$. Similarly, the vector $\theta_N^*$ introduced earlier, can be re-written as

$$\theta_N^* = \theta_N(\mathbf{z}_{i,N}, \mathbf{d}_{i,N}, \mathbf{x}_{J,N}^*) = \theta_N(\mathbf{z}_{i,N-1} - \Delta_M^\prime \mathbf{x}_{i,N-1}, (\mathbf{d}_{i,N-1} + \mathbf{k} \mathbf{x}_{i,N-1})e^{-\rho T}, \mathbf{z}_{i,N-1} - \Delta_M^\prime \mathbf{x}_{i,N-1})$$

$$= \phi_{N-1}(\mathbf{z}_{i,N-1}, \mathbf{d}_{i,N-1}, \mathbf{x}_{i,N-1}),$$

where we introduced $\phi_{N-1} = \mathbf{x}_{J,N-1}^\prime \mathbf{\Psi}_{\phi_{N-1}} \mathbf{x}_{i,N-1} + \mathbf{x}_{J,N-1}^\prime \mathbf{\Psi}_{\phi_{N-1}}$, a deterministic function of the state vectors at $N-1$ with the same quadratic properties as $\theta_N^*$. It follows that

$$J_{N-1} = \max_{\mathbf{x}_{J,N-1}} \mathbb{E}_{N-1} \left[ -e^{-\alpha(W_{N-2} + \theta_{N-1} + \phi_{N-1} - \mathbf{x}_{J,N-1} \Delta_M u_{N-1}) + (\theta_N^* - \mathbf{z}_{j,N-1} \Delta_M u_{N-1})} \right].$$

Lastly, replacing $\mathbf{u}_{i,N}$ by its recursive form $\mathbf{u}_{i,N} = \mathbf{u}_{i,N-1} + \epsilon_i \rho$, we obtain

$$J_{N-1} = \max_{\mathbf{x}_{J,N-1}} \mathbb{E}_{N-1} \left[ -e^{-\alpha(W_{N-2} + \theta_{N-1} + \phi_{N-1} - \mathbf{z}_{j,N-1} \Delta_M u_{N-1}) + (\theta_N^* - \mathbf{z}_{j,N-1} \Delta_M) \mathbf{e}_{i,N}} \right].$$

(41)

At this stage, we highlight several important properties: The expectation is conditional on the adapted filtration $\mathcal{F}_{N-1}$, implying that the only stochastic term is $\epsilon_i \rho$, which is normally distributed. Furthermore, the path-dependent term $\mathbf{x}_{J,N-1} \Delta_M u_{N-1}$ cancels out during the last operation. This will have important consequences on the structure of the optimal policy at $N-1$, as we will show in the first-order conditions.
Next, we separate the deterministic and stochastic terms by defining the functions

\[ V = W_{N-2} + \theta_{N-1} + \phi_{N-1} - z'_{i:N-1} u_{i:N-1}, \]  
\[ \tilde{V} = (z'_{i:N-1} - x'_{i:N-1} \Delta M) \epsilon_{i:N}. \]  

(42) \hspace{2cm} (43)

To summarize, we have shown that the value function at \( N-1 \) can be written as

\[ J_{N-1} = \max_{x_{i:N-1}} \mathbb{E}_{N-1} \left[ -e^{-\alpha (V - \tilde{V})} \right], \]

where by construction, \( V \) is a deterministic function, conditional on \( \mathcal{F}_{N-1} \), and \( \tilde{V} \) is normally distributed. Given these properties, and using the identity \( \mathbb{E}[e^{\tilde{V}}] = e^{\mathbb{E}[\tilde{V}] + \frac{1}{2} \text{Var}[\tilde{V}]} \) for any normal distributed variable \( \tilde{V} \), we have

\[ J_{N-1} = \max_{x_{i:N-1}} -e^{-\alpha \mathbb{E}_{N-1}[V - \tilde{V}] + \frac{\alpha^2}{2} \text{Var}_{N-1}[V - \tilde{V}]]. \]  

(44)

By monotonicity of the exponential, the optimal policy at \( N-1 \) can be obtained by solving the equivalent optimization problem given by

\[ x^*_{i:N-1} = \arg \max_{x_{i:N-1}} \mathbb{E}_{N-1} [V - \tilde{V}] - \frac{\alpha}{2} \text{Var}_{N-1}[V - \tilde{V}]. \]

Thus, we can see that at \( N-1 \), optimizing over an exponential utility is equivalent to optimizing over a mean-variance objective.

The next step is to gain some insights into the properties of the optimal control at \( N-1 \). For this, we will continue working with the mean-variance form. Following equations (42) and (43), the mean is simply

\[ \mathbb{E}_{N-1}[V - \tilde{V}] = W_{N-2} + \theta_{N-1} + \phi_{N-1} - z'_{i:N-1} u_{i:N-1}, \]

while the variance is

\[ \text{Var}_{N-1}[V - \tilde{V}] = \text{Var}_{N-1}[\tilde{V}] = \text{Var}_{N-1}[(z'_{i:N-1} - x'_{i:N-1} \Delta M) \epsilon_{i:N}] = (z'_{i:N-1} - x'_{i:N-1} \Delta M) (\tau \Sigma \epsilon) (z_{i:N-1} - \Delta'_M x_{i:N-1}). \]

We can see that the \( N-1 \) variance only depends on the state vector of remaining trades and the control at \( N-1 \). Combining the mean and variance expressions we have

\[ x^*_{i:N-1} = \arg \max_{x_{i:N-1}} W_{N-2} + \theta_{N-1} + \phi_{N-1} - z'_{i:N-1} u_{i:N-1} - \frac{1}{2} \alpha (z'_{i:N-1} - x'_{i:N-1} \Delta M) (\tau \Sigma \epsilon) (z_{i:N-1} - \Delta'_M x_{i:N-1}). \]

(45)

By construction, the objective in equation (45) is quadratic in the controls and the state, but is not necessarily concave in \( x_{i:N-1} \) for all possible parameter values of the problem. Concavity conditions follow by imposing
negative semidefiniteness of the matrix $\Psi_{\theta_{N-1}} + \Psi_{\phi_{N-1}} - \frac{1}{2} \alpha \tau \Delta M \Sigma \Delta' M$. Assuming concavity holds, the first order conditions at this stage give

$$0 = \nabla_{x_{N-1}} (\phi_{N-1} + \theta_{N-1}) + \alpha (z'_{N-1} - x'_{N-1} \Delta M) (\tau \Sigma \epsilon).$$

Looking at the system of equations obtained from the first-order conditions, it is clear that $x^*_i$ is not path-dependent. Indeed, there is no term which depends explicitly on the realization of the random walk $u_{i,N-1}$, at $N - 1$. Furthermore, since $\theta_{N-1}$ and $\phi_{N-1}$ are, by construction, quadratic functions of $x_{i,N-1}$ and only depend on the state vectors at $N - 1$, then $x^*_i$ will simply be a function of the two state vectors $z_{i,N-1}$ and $d_{i,N-1}$. We write the general form as

$$x^*_i = H_{i,N-1}(z_{i,N-1}, d_{i,N-1}), \quad (46)$$

where $H_{i,N-1}$ is a deterministic function of the state vectors at $N - 1$ whose exact expression is not relevant for the proof. It follows that plugging this back into $J_{N-1}$ will yield a value function which takes the form

$$J^*_i = -e^{-\alpha (W_{N-2} + \Theta_{N-1} - z'_{N-1} u_{N-1})},$$

where we have separated the exponential into a deterministic term:

$$\Theta^*_i = \Theta_{N-1}(z_{i,N-1}, d_{i,N-1}, x^*_{i,N-1})$$

$$= \Theta_{N-1}(z_{i,N-1}, d_{i,N-1}, H_{i,N-1}(z_{i,N-1}, d_{i,N-1}))$$

$$= \Theta^*_{N-1}(z_{i,N-1}, d_{i,N-1})$$

$$= \phi^*_{N-1} + \theta^*_{N-1} - \frac{1}{2} \alpha (z'_{N-1} - x'_{N-1} \Delta M) (\tau \Sigma \epsilon) (z_{N-1} - \Delta M x^*_{N-1}),$$

and a path-dependent term: $z'_{N-1} u_{N-1}$. We proceed to prove by induction that this separability property is conserved for all times, leading to optimal controls that are path-independent.

**General proof by induction.**

Let $J_n = J_n(z_n, d_n, W_{n-1}, n)$ be the value function at time $n$. The equivalent DP of problem (18) without inequality constraints, is given by

$$J_{n-1} = \max_{x_{n-1}} E_{n-1}[J^*_n]. \quad (47)$$

\(^1\)It is relatively straightforward to check that a non-empty set of parameters exists, for which concavity holds. A trivial case is when $\alpha = 0$ and there is no permanent and cross-impact, in which case, all matrices at each time step are diagonal by construction, with negative eigenvalues that are proportional to the inverse densities $1/q^a$ and $1/q^b$ of the ask and bid sides.
Assume that the value function and optimal policy at \( n \) take the following forms:

\[
J_n^* = -e^{-\alpha (W_{n-1} + \Theta_n^* - z_n^* u_n)},
\]

\[
x_n^* = H_n(z_n, d_n),
\]

where \( \Theta_n^* = \Theta_n^*(z_n, d_n) = \Theta_n(z_n, d_n, x_n^*) \). We need several properties to hold. The first is that if the value function has this form at time \( n \), it will lead to an optimal control \( x_{n-1}^* \) that is path-independent at time \( n - 1 \). The second is that at \( n - 1 \), \( J_{n-1} \) will conserve this separable form. This would also imply that \( x_{n-1}^* \) would be path-independent, and so forth. The third property is to check that this holds true at the boundary, which we have already confirmed through our previous analysis at times \( N \) and \( N - 1 \). The fourth and final property is to impose concavity of the objective at each time \( n \). Next, we look at an arbitrary time \( n - 1 \). From the induction assumption (48), we have

\[
J_n^* = -e^{-\alpha (W_{n-1} + \Theta_n^* - z_n^* u_n)},
\]

which after using Assumption 2 and equation (38), can be written as

\[
J_n^* = -e^{-\alpha \left(W_{n-2} + \theta_n - z_{n-1}^* u_{n-1} - (z_{n-1}^* - x_{n-1}^* \Delta) \epsilon_n \right)}.
\]

Using the recursive state equations we can express \( \Theta_n^* \) as a function of the states at \( n - 1 \).

\[
\Theta_n^* = \Theta_n^*(z_{n-1}, d_{n-1}) = \Theta_n(z_{n-1} - x_{n-1}, (d_{n-1} + \kappa x_{n-1}) e^{-\rho t})
\]

\[
= \Phi_n(z_{n-1}, d_{n-1}, x_{n-1})
\]

\[
= \Phi_{n-1}.
\]

so that

\[
J_n^* = -e^{-\alpha \left(W_{n-2} + \theta_n - \Phi_{n-1}^* - z_{n-1}^* u_{n-1} - (z_{n-1}^* - x_{n-1}^* \Delta) \epsilon_n \right)}.
\]

Next, we look at the dynamic programming equation (47). We have

\[
J_{n-1} = \max_{x_{n-1}} \mathbb{E}_{n-1} [J_n^*]
\]

\[
= \max_{x_{n-1}} \mathbb{E}_{n-1} \left[-e^{-\alpha \left(W_{n-2} + \theta_n - \Phi_{n-1}^* - z_{n-1}^* u_{n-1} - (z_{n-1}^* - x_{n-1}^* \Delta) \epsilon_n \right)}\right]
\]

\[
= \max_{x_{n-1}} -e^{-\alpha \mathbb{E}_{n-1} \left[W_{n-2} + \theta_n - \Phi_{n-1}^* - z_{n-1}^* u_{n-1} - (z_{n-1}^* - x_{n-1}^* \Delta) \epsilon_n \right]}
\]

\[
+ \frac{\alpha}{2} \mathbb{E}_{n-1} \left[W_{n-2} + \theta_n - \Phi_{n-1}^* - z_{n-1}^* u_{n-1} - (z_{n-1}^* - x_{n-1}^* \Delta) \epsilon_n \right].
\]


After taking the mean and the variance at \( n - 1 \), the optimal policy at this stage can be obtained by solving the equivalent mean-variance optimization problem

\[
x^*_n \ = \ \arg \max_{\mathbf{x}_n} W_{n-2} + \theta_{n-1} + \Phi_{n-1} - \mathbf{z}'_{n-1} \mathbf{u}_{n-1} \]

\[
- \frac{1}{2} \alpha (\mathbf{z}'_{n-1} - \mathbf{x}'_{n-1} \Delta_M) \tau \Sigma \epsilon (\mathbf{z}_{n-1} - \Delta'_M \mathbf{x}_{n-1}),
\]

which is quadratic in \( \mathbf{x}_{n-1} \) as well as in the state vectors at \( n-1 \). Concavity in \( \mathbf{x}_{n-1} \) is imposed by requiring that the matrix \( \Psi_{\theta_{n-1}} + \Psi_{\Phi_{n-1}} - \frac{1}{2} \alpha \tau \Delta_M \Sigma \epsilon \Delta'_M \) is negative semidefinite. The first order conditions will yield an optimal control which is clearly path-independent as the only term which explicitly depends on \( \mathbf{u}_{n-1} \) is not a function of \( \mathbf{x}_{n-1} \). Letting \( \mathbf{x}^*_n = \mathbf{H}_{n-1}(\mathbf{z}_{n-1}, \mathbf{d}_{n-1}) \) be the optimal solution at \( n - 1 \), we complete the induction proof by setting \( \Theta^*_n = \theta^*_n + \Phi^*_n - \frac{1}{2} \alpha (\mathbf{z}'_{n-1} - \mathbf{x}'^*_n \Delta_M) \tau \Sigma \epsilon (\mathbf{z}_{n-1} - \Delta'_M \mathbf{x}^*_n) \), which gives the form we need, namely

\[
J^*_n = -e^{-\alpha (W_{n-2} + \Theta^*_n - \mathbf{z}'_{n-1} \mathbf{u}_{n-1})}.
\]

We can thus conclude that the optimal control at the previous time step is path-independent and separability is preserved.

**Inequality constrained problem.**

To complete the proof, we still need to look at the case with inequality constraints on the control variables \( \mathbf{x}_n \geq 0 \). Let \( \mathbf{\nu}_n \) be the associated positivity multiplier vector at time \( n \). To establish path-independence for the optimal solution in this case, we need to show that both \( \mathbf{x}^*_n \) and \( \mathbf{\nu}_n \) are path-independent. The formal proof for this follows and is similar to the unconstrained problem. We can obtain the necessary insights by looking at the two last periods of the problem. At time \( N \), we have by definition \( 0 \leq \mathbf{x}^*_N \) and \( \mathbf{x}'_N \Delta_M = \mathbf{z}_N \). Looking at the problem at \( N - 1 \), and using equation (44), we have

\[
J_{N-1} = \max_{\mathbf{x}_{N-1}} -e^{-\alpha \mathbb{E}_{N-1}[V-\hat{V}]} + \frac{\alpha^2}{2} \text{Var}_{N-1}[V-\hat{V}]
\]

subject to \( \mathbf{x}_{N-1} \geq 0 \).

We can then write an equivalent maximization problem to the Problem (51), by taking the logarithm and plugging in the forms from equations (42) and (43). We obtain:

\[
\max_{\mathbf{x}_{N-1}} W_{N-2} + \theta_{N-1} + \phi_{N-1} - \mathbf{z}'_{N-1} \mathbf{u}_{N-1}
\]

\[
- \frac{1}{2} \alpha (\mathbf{z}'_{N-1} - \mathbf{x}'_{N-1} \Delta_M) (\tau \Sigma \epsilon) (\mathbf{z}_{N-1} - \Delta'_M \mathbf{x}_{N-1})
\]

subject to \( \mathbf{x}_{N-1} \geq 0 \).
To deal with the inequality constraint, we introduce the Lagrange positivity multiplier $\nu_{N-1}$. The problem then becomes:

$$
\max_{x_{N-1}} W_{N-2} + \theta_{N-1} + \phi_{N-1} - x'_{N-1} \Delta z_{N-1} - \frac{1}{2} \alpha (z'_{N-1} - x'_{N-1} \Delta M)(\tau \Sigma e)(z_{N-1} - \Delta_{N} M x_{N-1}) + \nu'_{N-1} x_{N-1}.
$$

Imposing concavity on the previous equation, the first order conditions at this stage give:

$$
0 = \nabla x_{N-1}(\phi_{N-1} + \theta_{N-1}) + \alpha (z'_{N-1} - x'_{N-1} \Delta M)(\tau \Sigma e) + \nu_{N-1},
$$

where, by definition of $\phi_{N-1}$ and $\theta_{N-1}$, the term $\nabla x_{N-1}(\phi_{N-1} + \theta_{N-1})$ is a deterministic linear function of $x_{N-1}$. This is also clearly the case for the second term $\alpha (z'_{N-1} - x'_{N-1} \Delta M)(\tau \Sigma e)$ which only depends on the control and state vectors at $N-1$ and the stationary covariance matrix. As there are no stochastic terms in the system of equations, we can conclude that the multipliers at $N-1$ (which can be calculated by considering the complementary slackness and dual feasibility conditions for each asset at $N-1$) are necessarily deterministic. More specifically, from equation (52), we can write the form of the optimal control at $N-1$ as

$$
x^*_N = H_{N-1}(z_{N-1}, d_{N-1}, \nu_{N-1}),
$$

where as before, the exact expression of $H_{N-1}$ is not relevant for the proof. Then, plugging this back into the objective function, it becomes clear that this new form does not affect the separability property. Indeed, the function $\Theta^*_{N-1}$ remains deterministic, but now depends on the multipliers:

$$
\Theta^*_{N-1} = \Theta_{N-1}(z_{N-1}, d_{N-1}, x^*_{N-1})
= \Theta_{N-1}(z_{N-1}, d_{N-1}, x^*_{N-1}, \nu_{N-1})
= \Theta^*_{N-1}(z_{N-1}, d_{N-1}, \nu_{N-1}).
$$

From here, we could proceed using the same induction arguments that we developed for the unconstrained problem to show that the first-order conditions at each time period lead to deterministic optimal controls and multipliers at all periods. We conclude that $\nu_n$ will preserve this path-independence property for all $n$.

Our path-independence result can be compared to other types of price impact models that have been developed in the literature. In particular, Almgren & Chriss (2000) and Huberman & Stanzl (2005) show that a similar static policy exists in their mean-variance framework. However, it is worth mentioning that this result is not generally robust to the type of noise process assumed in the model. For example, if we wanted to include serial correlation in our framework, we could show that this would lead to an optimal policy which is path-dependent (i.e., not static). In this case, we would need to develop a different solution methodology without being able to rely on static equivalence. In other words, the optimal policy would be adaptive. Other examples of adaptive optimal liquidation policies can be found in Lorenz & Almgren (2012).
Lemma 4 (Equivalent Wealth Formulation)

We proceed via verification, starting with the inferred form, and showing that by expansion, we obtain the desired expressions equivalent to (35) and (36). Expanding the wealth process we have

\[ W_n = -\begin{bmatrix} x^+ \\ x^- \end{bmatrix}^T \begin{bmatrix} D^a & -D^{ab} \\ -D^{ba} & D^b \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} - \begin{bmatrix} c^a \\ -c^b \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \]

\[ = -\left( x^+ D^a x^+ - x^+ D^{ab} x^- - x^+ D^{ba} x^+ + x^- D^b x^- \right) - (u + \frac{1}{2}s)x^+ + (u - \frac{1}{2}s)x^- . \]

Focusing on the ask side, we can show after some algebra that

\[ x^+ D^a x^+ = x_1^+ (D_{11}^a x_1^+ + D_{12}^a x_2^+ + \ldots) + \ldots + x_M^+ (D_{M1}^a x_1^+ + D_{M2}^a x_2^+ + \ldots), \]

where expanding each term, we have

\[ x_i^+ D_{ii}^a x_i^+ = \sum_{n=0}^N x_{i,n}^+ \left( \frac{1}{2q_i^+} x_{i,n}^+ + \sum_{k=1}^n \lambda_i x_{i,k-1}^+ + \left( \frac{x_{i,k-1}^+}{q_i^+} - \lambda_i x_{i,k-1}^+ \right) e^{-\tau \rho_i^+(n-k+1)} \right). \]

Similarly, for the cross terms we have

\[ x_i^+ D_{ij}^a x_j^+ = \sum_{n=0}^N x_{i,n}^+ \left( \sum_{k=1}^n \lambda_{ij} x_{j,k-1}^+ - \lambda_{ij} x_{j,k-1}^+ e^{-\tau \rho_i^+(n-k+1)} \right). \]

Combining the above expressions and summing over \( i \) and \( j \), we obtain

\[ x^+ D^a x^+ = x_1^+ (D_{11}^a x_1^+ + D_{12}^a x_2^+ + \ldots) + \ldots + x_M^+ (D_{M1}^a x_1^+ + D_{M2}^a x_2^+ + \ldots) \]

\[ = \sum_{i=1}^M \sum_{n=0}^N x_{i,n}^+ \left[ \frac{1}{2q_i^+} x_{i,n}^+ + \sum_{k=1}^n \lambda_{ij} x_{j,k-1}^+ + \sum_{k=1}^n \left( \frac{x_{i,k-1}^+}{q_i^+} - \sum_{j} \lambda_{ij} x_{j,k-1}^+ \right) e^{-\tau \rho_i^+(n-k+1)} \right]. \]

Furthermore, we also need to include the cross-terms coming from the opposite side of the book:

\[ x^+ D^{ab} x^- = \sum_{i=1}^M \sum_{n=0}^N x_{i,n}^+ \left[ \sum_{k=1}^n \sum_{j} \lambda_{ij} x_{j,k-1}^- - \sum_{k=1}^n \sum_{j} \lambda_{ij} x_{j,k-1}^- e^{-\tau \rho_i^+(n-k+1)} \right]. \]

Lastly, we also provide the equivalence for the linear terms which is straightforward:

\[ (c^n)^T x^+ = (u + \frac{1}{2}s)^T x^+ = \left( \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix} + \frac{1}{2} \begin{bmatrix} s_1 \\ \vdots \\ s_M \end{bmatrix} \right)^T x^+ \]

\[ = \sum_i \sum_n (u_{i,n} + \frac{1}{2}s_i) x_{i,n}^+. \]
By identification, we recover the complete form of the ask-side price process (the bid-side is derived in a similar way). The form for the costs follows immediately.

**Equations (20) and (21) (Expectation and Variance)**

Following Proposition 1, the optimal control vector $x$ is deterministic. Therefore we can remove it from the operator and take the expectation directly on the linear term in equation (19).

$$E[W_N] = E[-x'Dx - \tilde{c}'x] = -x'Dx - E[\tilde{c}'x].$$

After some algebra, the expectation over the stochastic linear term can be expressed as

$$E[\tilde{c}'x] = u_0'1'[\Delta_K'x + \frac{1}{2}s'[\Delta_K']_+ x],$$

where $u_0 = [u_{1,0}; \ldots; u_{M,0}]$ and $u_0'1' = [u_{1,0}1_{N+1}; \ldots; u_{M,0}1_{N+1}]'$. The expected wealth then becomes

$$E[W_N] = -x'Dx - u_0'1'[\Delta_K'x - \frac{1}{2}s'[\Delta_K']_+ x].$$

We can show that the variance term reduces to

$$\text{Var}[W_N] = \text{Var}[-x'Dx - \tilde{c}'x] = \text{Var}[(u' + \frac{1}{2}s')x^+ - (u' - \frac{1}{2}s')x^-] = \text{Var}[u'[\Delta_K'x].$$

Which gives $\text{Var}[W_N] = x'\Delta_K\Sigma u[\Delta_K']x$. ■

**Proposition 2 (Equivalent Quadratic Program)**

Combining the equations we obtain for the mean and variance, the problem (18) can be written as

$$\text{maximize } x \geq 0 - x'Dx - u_0'1'[\Delta_K'x - \frac{1}{2}s'[\Delta_K']_+ x - \frac{1}{2}\alpha x'\Delta_K\Sigma u[\Delta_K']x$$

subject to $1'[\Delta_K'x = z_0$.}

The above problem can be equivalently written as a minimization problem over the risk-adjusted execution shortfall (i.e., net execution cost). The execution shortfall is defined as the difference between the pre-execution market value of the portfolio ($W_0$), and the expected post-execution wealth ($\mu W_N$), i.e., it is equal to $W_0 - \mu W_N$, where $W_0 = -u_0'z_0 = -u_0'(1'[\Delta_K'x)$ is constant and can thus be added to the objective function without affecting the optimal solution. The problem then becomes

$$\text{minimize } x \geq 0 - u_0'1'[\Delta_K'x + x'Dx + u_0'1'[\Delta_K'x + \frac{1}{2}s'[\Delta_K']_+ x + \frac{1}{2}\alpha x'\Delta_K\Sigma u[\Delta_K']x$$

subject to $1'[\Delta_K'x = z_0$.}
Let \( D^* = D + \frac{1}{2} \alpha \Delta_K \Sigma u \Delta'_K \). Since we have
\[
x'D^*x = x' \left( \frac{D^* + D'^*}{2} \right) x,
\]
we set the symmetric form \( \frac{1}{2} \tilde{D} = \left( \frac{D^* + D'^*}{2} \right) \). So finally, the problem (18) is equivalent to
\[
\begin{align*}
& \text{minimize} \quad \frac{1}{2} x' \tilde{D} x + c' x \\
& \text{subject to} \quad 1' \Delta'_K x = z_0,
\end{align*}
\]
where \( c' = \frac{1}{2} s' \Delta'_K \) and \( \tilde{D} = (D + \frac{1}{2} \alpha \Delta_K \Sigma u \Delta'_K) + (D + \frac{1}{2} \alpha \Delta_K \Sigma u \Delta'_K)' \).

**References**

