

**A Comparison of the Mean-Variance-Leverage  
Optimization Model and the Markowitz General  
Mean-Variance Portfolio Selection Model**

**Bruce I. Jacobs and Kenneth N. Levy**

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Bruce I. Jacobs and Kenneth N. Levy are Principals of Jacobs Levy Equity Management.

Jacobs Levy Equity Management  
100 Campus Drive, P.O. Box 650  
Florham Park, New Jersey 07932-0650  
Tel: 973-410-9222  
Email: [bruce.jacobs@jlem.com](mailto:bruce.jacobs@jlem.com)

## **Abstract**

The mean-variance-leverage (MVL) optimization model (Jacobs and Levy [2012, 2013]) tackles an issue not dealt with by the mean-variance optimization inherent in the general mean-variance portfolio selection model (GPSM) — that is, the impact on investor utility of the risks that are unique to using leverage. Relying on leverage constraints with a conventional GPSM, as is commonly done today, is unlikely to lead to the portfolio offering a leverage-averse investor the highest utility. But investors can use the MVL model to find optimal portfolios that balance expected return, volatility risk, and leverage risk. The MVL model has intuitive appeal and offers straightforward implementation for portfolio selection. In contrast, practical use of a broader application of GPSM, as suggested by Markowitz [2013], is dependent on successful future development of a stochastic margin-call model (SMCM).

INVITED EDITORIAL COMMENT

# **A Comparison of the Mean-Variance-Leverage Optimization Model and the Markowitz General Mean-Variance Portfolio Selection Model**

BRUCE I. JACOBS and KENNETH N. LEVY

In response to the mean-variance-leverage optimization model (MVL model) proposed in Jacobs and Levy [2013], Markowitz [2013] suggested a broader application of the general mean-variance portfolio selection model (GPSM) (Markowitz [1959], Markowitz and Todd [2000]) that differs from conventional usage of the GPSM.

We developed the MVL model because the mean-variance (MV) optimization inherent in the conventional GPSM does not consider components of risk that are unique to using leverage. These include the risks and costs of margin calls—which can force borrowers to liquidate securities at adverse prices due to illiquidity—losses exceeding the capital invested, and the possibility of bankruptcy.

The MVL model provides practical insights on investors' aversion to leverage, as well as a methodology that is straightforward and ready to implement. In contrast, practical implementation of the broader GPSM, and economic insights it may provide, are dependent on the successful future development of a stochastic margin call model (SMCM).

Markowitz [2013] notes that our MVL model uses part of the apparatus of the conventional GPSM. Like the GPSM, the MVL model allows for shorts, leveraged longs, and constraints on both individual

securities and the portfolio. Our joint previous work with Markowitz addressed leverage-constrained GPSM optimization (Jacobs, Levy, and Markowitz [2005, 2006]). There are, however, several aspects of the MVL model that differentiate it from both the conventional and broader GPSM.

### **LEVERAGE RISK—A THIRD DIMENSION**

In Jacobs and Levy [2013], we specified an MVL utility function that is broader than the MV utility function used in the conventional GPSM. We augmented the MV utility function with a leverage-aversion term, which is the third term in the following equation:

$$U = \alpha_p - \frac{1}{2\tau_v} \sigma_p^2 - \frac{1}{2\tau_L} \sigma_r^2 \Lambda^2 \quad (1)$$

where  $\alpha_p$  is the expected active return (relative to the benchmark) of the leveraged portfolio;  $\sigma_p^2$  is the variance of the leveraged portfolio's active return;  $\tau_v$  is the investor's risk tolerance with respect to the variance of the portfolio's active return, which we will refer to as volatility tolerance;  $\sigma_r^2$  is the variance of the leveraged portfolio's total return;  $\tau_L$  is the investor's leverage tolerance; and  $\Lambda$  is the sum of the absolute values of the portfolio holding weights minus 1:<sup>1</sup>

$$\Lambda = \sum_{i=1}^N |h_i| - 1 \quad (2)$$

where  $h_i$  is the portfolio holding weight of security  $i$ , and  $N$  is the number of securities in the selection universe. The active weight,  $x_i$ , of security  $i$  is equal to  $h_i$  minus its benchmark weight,  $b_i$ :

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<sup>1</sup> Leverage is measured in excess of 1, that is, in excess of 100% of net capital.

$$x_i = h_i - b_i \tag{3}$$

The MVL model extends the MV model to incorporate leverage risk as a third dimension. Similarly, Markowitz [2013] introduces a third dimension to the GPSM in the form of short-run portfolio variance,  $V^S$ . It is important to note that without a third dimension, the unique risks of leverage would not be represented.

The MVL leverage-aversion term in Equation (1) is a proxy for leverage risk.<sup>2</sup> This assumes that the risks of leverage rise with the product of the variance of the leveraged portfolio's total return and the square of the portfolio's leverage. By multiplying the square of the portfolio leverage by the variance of total return, we obtain a measure of the leverage risk's severity. For instance, leveraging highly volatile stocks entails more leverage risk (or margin-call risk) than leveraging stable stocks.

Also, by using the variance of total return (the square of the standard deviation of total return) to represent portfolio volatility, and by squaring leverage, the model recognizes that margin-call risk increases more rapidly as volatility and/or leverage increases. An advantage of the MVL model proxying leverage risk, rather than modeling margin calls explicitly, is that the MVL's leverage-aversion term is easily calculated. It requires only expected total portfolio volatility, the level of portfolio leverage, and the investor's leverage tolerance.

In contrast, Markowitz's expanded model requires the development of an SMCM, to be used along with measures of short-run portfolio variance,  $V^S$ , and long-run portfolio variance,  $V^L$ . These portfolio-variance measures require defining the periods referred to as short-run and long-run. The SMCM would be used to calculate: 1) the probability that there

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<sup>2</sup> We use the terms "tolerance" and "aversion" with the understanding that they are the inverse of each other.

will be one or more margin calls on a portfolio; 2) the portfolio return variance conditional on the occurrence of one or more margin calls; and 3) the reduction in the portfolio's expected return conditional on the occurrence of one or more margin calls.

To accurately incorporate all these risks is a formidable problem that has yet to be solved. As Markowitz [2013] notes, only when one has developed a good SMCM can the broader GPSM find realistic efficient portfolios.

## **QUARTIC VERSUS QUADRATIC OPTIMIZATION**

The MVL approach differs from both GPSM approaches in the computation required. The GPSM is a quadratic mathematical problem (Markowitz [1959]). All terms in the utility function are linear or quadratic. A linear term is directly proportional to the active weight variable  $x_i$ . A quadratic term is proportional to the square of the active-weight variable  $x_i$ , including second-order cross-product terms. The application of GPSM as proposed in Markowitz [2013] would require using a quadratic solver repeatedly to create many mean-variance efficient portfolios, then computing the adjusted mean and variance of these portfolios by running each portfolio through the SMCM to make adjustments for margin-call risk.

The MVL model, in contrast, is a quartic mathematical problem. The leverage-aversion term in the MVL utility function (representing the product of the square of the standard deviation of the leveraged portfolio's total return and the square of the portfolio's leverage) is quartic in the active weight variable  $x_i$ , including fourth-order cross-product terms. Hence, the MVL cannot be solved directly with quadratic optimization. A solution method is fixed-point iteration that applies a

quadratic solver iteratively in order to provide the optimal portfolio (Jacobs and Levy [2012, 2013]).

## **PRACTICAL INSIGHTS FROM THE MVL OPTIMIZATION MODEL**

The broader GPSM and the MVL model differ in the underlying economic intuition and insights that they provide. The GPSM approach proposed by Markowitz [2013] is unable to yield practical economic insights until an accurate SMCM has been developed. In contrast, the MVL approach yields practical insights that can help investors understand the importance of leverage in selecting optimal portfolios.

A comparison of the MVL model and the conventional GPSM shows that the MV model is a special case of the MVL model. As the investor's tolerance for the unique risks of leverage approaches zero, the investor has an infinite aversion to leverage and the optimizer forces the portfolio's leverage level to zero. Because there is no leverage present, the MVL model reduces to the traditional long-only MV model. At the other extreme, as the investor's tolerance for the unique risks of leverage approaches infinity, the investor has no aversion to leverage, the leverage term in the MVL model is multiplied by zero leverage aversion, and that term drops out of the MVL utility function. Again, the MVL model reduces to the MV model.

The MV model, used with a constraint enforcing zero leverage, therefore implies that the investor has an infinite aversion to the unique risks of leverage, or zero leverage tolerance. Used without a leverage constraint, the MV model implies that the investor has zero aversion to the unique risks of leverage, or infinite leverage tolerance. Note that, although we observe zero leverage tolerance in practice—some investors are averse to any borrowing—infinite leverage tolerance seems contrary

to investor behavior, because it can give rise to extreme levels of leverage, in the absence of upper bounds on individual security holdings (Jacobs and Levy [2013, 2014]).

To avoid excessive leverage, the common practice today is to constrain it at some level.<sup>3</sup> For an investor who is averse to leverage, using the conventional mean-variance utility function and optimizing with a leverage constraint is unlikely to lead to the portfolio offering the highest utility. This is because a leverage constraint denies the investor the ability to balance the economic tradeoffs between expected portfolio return, portfolio volatility risk, and portfolio leverage risk (Jacobs and Levy [2014]).

Using the MVL model, there is a different mean-variance efficient frontier for any given level of investor leverage tolerance. Rather than one conventional MV efficient frontier, there are numerous frontiers, which fan out into an efficient region (Jacobs and Levy [2013]). For a particular level of leverage tolerance, there exists a unique mean-variance efficient frontier within the efficient region. Investors select optimal portfolios from the efficient frontier that correspond to their preferred leverage tolerances. Investors may prefer a lower frontier to a higher frontier because they may have a more moderate level of leverage tolerance, even though a lower frontier offers a lower level of expected return at each expected volatility level than does a higher frontier.

It may appear that this choice of a lower frontier contradicts the basic tenets of mean-variance portfolio theory. However, the investor's preference for a lower frontier, despite its lower expected returns, reflects the investor's aversion to the unique risks associated with the higher

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<sup>3</sup> There may be some absolute leverage level that an MV investor is required not to exceed. A certain level of leverage can be specified as an equality constraint or, alternatively, leverage can be constrained not to exceed a certain level, as an inequality constraint. A leverage constraint can also be imposed in MVL optimization.



frontier's higher leverage. The volatility-averse MV investor accepts a lower expected return in exchange for less volatility risk, and the leverage-averse MVL investor accepts a lower expected return in exchange for less volatility risk and less leverage risk. Put simply, leverage aversion affects portfolio choice.

## **CONCLUSION**

The MVL model provides many practical insights, and implementation for portfolio selection is straightforward. In contrast, practical use of the broader GPSM, as suggested by Markowitz, is dependent on successful future development of an SMCMM.

The MVL model allows the investor to consider both volatility tolerance and leverage tolerance in selecting optimal portfolios from either a three-dimensional mean-variance-leverage efficient surface (Jacobs and Levy [2012, 2014]), or a two-dimensional mean-variance efficient region (Jacobs and Levy [2013]). The MVL formulation provides a formal model that has intuitive appeal.

Relying on leverage constraints with a conventional GPSM, as is commonly done today, is unlikely to lead to the portfolio offering a leverage-averse investor the highest utility. However, investors can use the MVL model to find optimal portfolios that balance expected return, volatility risk, and leverage risk.

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