Risk Premia, Volatilities, and Sharpe Ratios in a Non-Linear Term Structure Model

Peter Feldhütter† Christian Heyerdahl-Larsen‡
London Business School London Business School

Philipp Illeditsch§
The Wharton School

May 13, 2013

Abstract

In this paper we propose an expansion of Gaussian term structure models where the short rate and market prices of risk are non-linear in the state variables. We provide closed-form solutions for bond prices and since the latent factors are Gaussian the expanded model is as tractable as the Gaussian model. We estimate a three-factor expanded model and find that the model matches the time variation in both expected excess returns and yield volatilities of U.S. Treasury bonds. Comparing Sharpe ratios in the Gaussian and expanded model, the expanded model implies that Treasury bonds are more attractive investments in periods with low volatility. A significant part of expected excess returns in the expanded model is not spanned by the cross section of yields. This suggests that more information than previously thought is contained in the yield curve, but in a non-linear way.

Keywords: Affine term structure models, non-linear term structure models, time-varying term premiums, time-varying investment opportunities, stochastic volatility, Sharpe ratios, hidden factors.

JEL Classification: D51, E43, E52, G12.

*We would like to thank seminar participants at the London Business School, the Wharton School, and the University of Melbourne for helpful comments and suggestions.
†London Business School, Regent’s Park, London, NW1 4SA, pfeldhutter@london.edu
‡London Business School, Regent’s Park, London, NW1 4SA, cheyerdahlarsen@london.edu
§The Wharton School, University of Pennsylvania, pille@wharton.upenn.edu
I Introduction

Investments in any market require an assessment of both risk and reward in terms of expected excess returns and volatility of expected excess returns. In the U.S. Treasury bond market, Gaussian models are typically used to describe time-varying investment opportunities because of their analytic tractability. However, since volatility is constant in Gaussian models we can only learn about reward and not risk. To overcome this we propose a simple expansion of Gaussian term structure models where the short rate and market prices of risk are non-linear in the state variables. We provide closed-form solutions for bond prices and since the latent factors are Gaussian our expanded model is as tractable as the Gaussian model.

We expand a three-factor Gaussian model and estimate both the Gaussian and expanded model using U.S. Treasury data from 1952 to 2011. In a large part of the sample period the expanded and Gaussian model agree on expected excess returns, but in the early eighties the expanded model predicts historically high excess returns while the Gaussian model predicts excess returns close to the sample average. This period witnessed the highest realized excess returns during the sample period consistent with the prediction of the expanded model. For forecast horizons beyond one year the expanded model predicts excess returns better in-sample than the Gaussian model and even the single factor model of Cochrane and Piazzesi (2005). Thus, the non-linear relation between yields and excess returns in the expanded model is important for matching periods of high expected excess returns. This finding is relevant not only for term structure modeling but also for the common approach of predicting excess returns in vector-autoregressive (VAR) models. Joslin, Singleton, and Zhu (2011) show that conditional expected excess returns in a Gaussian model without parameter restrictions are identical to those from an unrestricted VAR model. Thus, a three-factor VAR with yields as factors would also miss the rise in expected excess
returns during the early eighties.

We also study whether our expanded Gaussian model captures the time variation and persistence of yield volatilities in the U.S. Treasury bond market. The average correlation between model-implied and actual volatility - proxied by an EGARCH model - is 71.5%, so the expanded model matches volatility well. Importantly, the expanded model matches the variation in bond volatility for both short and long maturity bonds and captures the spike in volatility in the early eighties. In contrast, affine models with stochastic volatility have too low volatility variation in short maturity bonds and cannot capture volatility spikes (Jacobs and Karoui (2009)). This shows that the non-linearity introduced in the expanded model is important for capturing the time variation in the volatility of yields.

Affine models have difficulty jointly matching expected excess returns and return volatility (Dai and Singleton (2002)). As a consequence, studies of the risk-return tradeoff in the U.S. Treasury market usually sacrifice matching return volatility and focus on Gaussian models.\footnote{Examples include Sangvinatsos and Wachter (2005), Duffee (2010a), and Joslin, Singleton, and Zhu (2011).} Since our expanded three-factor models matches both the time variation in excess returns and the volatility of excess returns well, we can use the model to better understand the risk-return tradeoff in the U.S. Treasury bond market. We compare model-implied conditional Sharpe ratios from the Gaussian model to its expanded counterpart. Both models predict similar Sharpe ratios during the high volatility regime in the early eighties while the expanded model predicts higher Sharpe ratios during low volatility regimes. The reason is that the constant volatility in the Gaussian model is fitted to average volatility in the sample period, and the Gaussian model therefore overestimates volatility in low-volatility periods. Hence, U.S. Treasury bonds are more attractive investments in low-volatility periods than what is implied by a standard Gaussian model.
There is growing evidence that macro variables have predictive power for excess returns and some of this predictive power is orthogonal to the yield curve. This finding motives Gaussian "hidden factor" models where one or more factors determine excess returns but are partially unrelated to the cross-section of yields. Our expanded model highlights an alternative channel through which excess returns are imperfectly correlated with the yield curve. In the model expected excess returns are non-linearly related to yields and therefore expected excess returns are not spanned by the cross-section of yields. Specifically, we find that between 14% and 29% of the variation in expected excess returns can not be explained by variations in yields. Furthermore, the unspanned part of expected excess returns is positively related to inflation. Thus, non-linearity is likely to be important to fully understand the relation between the unspanned part of expected excess returns and macro variables. The unspanned part appears in the three-factor expanded model without further restrictions on parameters or yield observation errors. In contrast, Duffee (2011) finds that at least five latent factors along with yield measurement errors are needed in a Gaussian model to generate a partially hidden factor. Alternatively, Joslin, Priebsch, and Singleton (2012) restrict parameters such that hidden factors appear and estimate a five-factor macro-finance Gaussian model. Unless economically motivated restrictions are imposed on parameters, a five-factor model leads to huge Sharpe ratios and thus to overfitting (Duffee (2010a)).

Now to the specifics of the expanded Gaussian term structure model. The standard procedure in the term structure literature is to specify the short rate and the market prices of risk as a function of the state variables. Instead, we model the functional form of the stochastic discount factor directly by multiplying the stochastic discount factor from an affine Gaussian term structure model with the term \( (1 + \gamma e^{-\beta X})^\alpha \) where \( X \) is a Gaussian state vector and \( \alpha, \beta, \) and \( \gamma \) are parameters. This functional

\footnote{Examples include Ludvigson and Ng (2009), Cooper and Priestley (2009), Cieslak and Povala (2010), Duffee (2011), Joslin, Priebsch, and Singleton (2012), and Chernov and Mueller (2012).}
form of the stochastic discount factor arises in many equilibrium models in the literature as we show in Appendix B. In such models the stochastic discount factor can be decomposed into a weighted average of different representative agent models. Importantly, the weights on the different models are time-varying and this is a source of time-varying risk premia and volatility of bonds. Most of these models impose strong assumptions on the dynamics of the state variables and the correlation between them. Our reduced form approach does not impose any such restriction, and thus allows a more flexible setup than most equilibrium models.

The rest of the paper is organized as follows. Section II describes the model. Section III estimates the model and Section IV presents the empirical results. Section V concludes.

II The Model

In this section we present a model of the term structure of interest rates. Uncertainty is represented by a $d$-dimensional Brownian motion $W(t) = (W_1(t), ..., W_d(t))'$. There is a $d$-dimensional Gaussian state vector $X(t)$ that follows the dynamics

$$dX(t) = \kappa (\bar{X} - X(t)) \, dt + \Sigma \, dW(t),$$

where $\bar{X}$ is $d$-dimensional and $\kappa$ and $\Sigma$ are $d \times d$-dimensional.

II.A The Stochastic Discount Factor

We assume that there is no arbitrage and hence there exists a strictly positive state price density or stochastic discount factor $M(t)$. Let $\gamma$ denote a non-negative constant and $M_0(t)$ a strictly positive stochastic process. The stochastic discount factor is
defined as
\[ M(t) = M_0(t) \left( 1 + \gamma e^{-\beta X(t)} \right)^\alpha, \]
(2)
where \( \beta \in \mathbb{R}^d \) and \( \alpha \in \mathbb{N} \).

Equation (2) is a key departure from standard term structure models (Vasicek (1977), Cox, Ingersoll, and Ross (1985), Duffie and Kan (1996), and Dai and Singleton (2000)). Rather than specifying the short rate and the market price of risk, which in turn pins down the state price density, we specify the functional form of the state price density directly. As Appendix B shows the functional form of \( M(t) \) arises in several equilibrium models in the literature.

To keep the model comparable to the existing literature on Gaussian term structure models we introduce a base model for which \( M_0(t) \) is the stochastic discount factor. The actual state price density \( M(t) \) collapses to the base model when \( \alpha = 0 \). To complete the model we assume that the dynamics of \( M_0(t) \) are
\[
\frac{dM_0(t)}{M_0(t)} = -r_0(t)dt - \Lambda_0(t)'dW(t),
\]
(3)
where \( r_0(t) \) and \( \Lambda_0(t) \) are affine functions of the state vector \( X(t) \). Specifically,
\[
r_0(X) = \rho_{0,0} + \rho_{0,X}X, \]
(4)
\[
\Lambda_0(X) = \lambda_{0,0} + \lambda_{0,X}X,
\]
(5)
where \( \rho_{0,0} \) is a scalar, \( \rho_{0,X} \) and \( \lambda_{0,0} \) are \( d \)-dimensional vectors, and \( \lambda_{0,X} \) is a \( d \times d \)-dimensional matrix. It is well known that bond prices in the base model (\( \alpha = 0 \)) belong to the class of essential affine term structure models (Duffee (2002) and Dai and Singleton (2002)). We now provide closed form solutions for bond prices for all \( \alpha \in \mathbb{N} \).
II.B Closed-Form Bond Pricing

Let $P^{(\tau)}(t)$ denote the price at time $t$ of a zero-coupon bond that matures in $\tau$ years. Specifically,

$$P^{(\tau)}(t) = \mathbb{E}_t \left[ \frac{M(t+\tau)}{M(t)} \right]. \quad (6)$$

We show in the next theorem that the price of a bond is a weighted average of bond prices in artificial economies that belong to the class of essential affine Gaussian term structure models.

**Theorem 1.** The price of a zero-coupon bond that matures in $\tau$ years is

$$P^{(\tau)}(t) = \sum_{n=0}^{\alpha} w_n(t) P_n^{(\tau)}(t), \quad (7)$$

where

$$P_n^{(\tau)}(t) = e^{A_n^{(\tau)} - B_n^{(\tau)}'X(t)}, \quad (8)$$

$$w_n(t) = \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right)^n e^{-n\beta'X(t)} \left( 1 + \gamma e^{-\beta'X(t)} \right)^{\alpha}. \quad (9)$$

The coefficient $A_n^{(\tau)}$ and the $d$-dimensional vector $B_n^{(\tau)}$ solve the ordinary differential equations

$$\frac{dA_n^{(\tau)}}{d\tau} = \frac{1}{2} B_n^{(\tau)} \Sigma \Sigma' B_n^{(\tau)} - B_n^{(\tau)}' \left( \kappa \bar{X} - \Sigma \lambda_n,0 \right) - \rho_n,0, \quad A_n^{(\tau)}(0) = 0, \quad (10)$$

$$\frac{dB_n^{(\tau)}}{d\tau} = - (\kappa + \Sigma \lambda_n,X)' B_n^{(\tau)} + \rho_n,X, \quad B_n^{(\tau)}(0) = 0_d, \quad (11)$$
where

\begin{align*}
\rho_{n,0} &= \rho_{0,0} + n\beta'\kappa \bar{X} - n\beta'\Sigma\lambda_{0,0} - \frac{1}{2}n^2\beta'\Sigma'\Sigma\beta, \\
\rho_{n,X} &= \rho_{0,X} - n\kappa'\beta - n\lambda'_{0,X}\Sigma'\beta, \\
\lambda_{n,0} &= \lambda_{0,0} + n\Sigma'\beta, \\
\lambda_{n,X} &= \lambda_{0,X}.
\end{align*}

(12) (13) (14) (15)

The proof of this theorem is provided in Appendix A. The intuition for the proof is as follows. Applying the Binomial Theorem to the state price density given in equation (2) leads to a finite sum of exponentials. Each summand can be interpreted as a stochastic discount factor in an artificial economy.\(^3\) Bond prices in these artificial economies belong to the class of essential affine term structure models and hence the bond price is a weighted average of these artificial bond prices. Equation (9) shows that the weights are non-linear functions of the state vector \(X\).

II.C The Short Rate and the Price of Risk

Applying Ito’s lemma to equation (2) leads to the dynamics of the stochastic discount factor:

\[
d\frac{M(t)}{M(t)} = -r(t)\,dt - \Lambda(t)'dW(t),
\]

where both the short rate \(r(t)\) and the market price of risk \(\Lambda(t)\) are non-linear functions of the state vector \(X(t)\) given in equations (17) and (18), respectively. Define

\(^3\)Similar expansions of the stochastic discount factor appear in Yan (2008), Dumas, Kurshev, and Uppal (2009), Bhamra and Uppal (2010), and Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2013).
\( s(X) = \frac{1}{1 + \gamma e^{-\beta'X}} \) and note that \( s(X) \in (0, 1) \). The short rate is given by

\[
\begin{align*}
    r(t) &= r_0(t) + \alpha (1 - s(t)) \beta' \kappa (\bar{X} - X(t)) - \alpha (1 - s(t)) \beta' \Sigma \Lambda_0(t) \\
    &\quad - \frac{\alpha}{2} (1 - s(t)) (\alpha (1 - s(t)) + s(t)) \beta' \Sigma' \Sigma' \beta.
\end{align*}
\] (17)

Our model allows the short rate to be non-linear in the state variables (an important property as highlighted by Chan, Karolyi, Longstaff, and Sanders (1992), Ait-Sahalia (1996a), Ait-Sahalia (1996b), Stanton (1997), Chapman and Pearson (2000), and Jones (2003)) without losing the tractability of closed form solutions of bond prices and a Gaussian state space.\(^4\)

The \( d \)-dimensional market price of risk is given by

\[
\Lambda(t) = \Lambda_0(t) + \alpha (1 - s(t)) \Sigma' \beta.
\] (18)

From equation (18) we can see that even if the market prices of risk in the base model are constant, the market prices of risks in the general model are stochastic due to variations in \( s(t) \). When \( s(t) \) approaches one, then the market prices of risk approaches \( \Lambda_0(t) \) and when \( s(t) \) approaches zero the market prices of risk approaches \( \Lambda_0(t) + \alpha \Sigma' \beta \).

\section*{II.D Expected Return, Volatility, and Sharpe Ratio}

We know that the bond price is a weighted average of exponential affine bond prices (see equation (7)). Hence, variations of instantaneous bond returns are due to variations in the artificial bond prices \( P^{(r)}_n(t) \) given in equation (8) and due to variations in the weights \( w_n(t) \) given in equation (9). Specifically, the dynamics of the bond

\(^4\)Jermann (2013) studies the nominal term structure of interest rates in a production economy and obtains a short rate and prices of risk that are non-linear in the state variables but he does not get closed form solutions for bond prices.
price \( P^{(\tau)}(t) \) are

\[
\frac{dP^{(\tau)}(t)}{P^{(\tau)}(t)} = \left( r(t) + e^{(\tau)}(t) \right) \, dt + v^{(\tau)}(t)' \, dW(t),
\]  

(19)

where \( e^{(\tau)}(t) \) denotes the instantaneous expected excess return and \( v^{(\tau)}(t) \) denotes the local volatility vector of a zero-coupon bond that matures in \( \tau \) years.

Let \( \omega^{(\tau)}_n(t) \) denote the contribution of each artificial exponential affine bond price to the total bond price. Specifically,

\[
\omega^{(\tau)}_n(t) = \frac{P^{(\tau)}_n(t)w_n(t)}{P^{(\tau)}(t)}.
\]  

(20)

Then the local volatility of the bond is given by

\[
v^{(\tau)}(t) = -\sum_{\alpha} \left( \sum_{n=0}^{\alpha} \omega^{(\tau)}_n(t)B_n^{\alpha}(\tau) + \beta \left( \sum_{n=0}^{\alpha} n \omega^{(\tau)}_n(t) - \alpha(1 - s(t)) \right) \right),
\]  

(21)

where the first term represent the risk coming from variations in \( P^{(\tau)}_n(t) \) and the last two terms represent the risk coming from variations in the weights \( w_n(t) \).

The instantaneous expected return of the bond is

\[
e^{(\tau)}(t) = \Lambda(t)'v^{(\tau)}(t).
\]  

(22)

Variations in expected excess returns are driven by variations in the quantities of risk \( v^{(\tau)}(t) \) and the prices of risk \( \Lambda(t) \).

\footnote{The local volatility \( v^{(\tau)}(t) \) goes to zero when \( \tau \) goes to zero because \( \sum_{n=0}^{\alpha} n w_n(t) = \alpha(1 - s(t)) \).}
II.E Simple Example

In this section we provide a simple example in which the base model is a one factor completely affine Vasicek model and $\alpha = 1$. We show how to determine the bond price in closed form and provide a simple expression for the short rate, the market price of risk, and the mean and volatility of the instantaneous bond return.

The stochastic discount factor is defined as

$$M(t) = M_0(t) \left(1 + \gamma e^{-\beta x(t)}\right),$$

where $x(t)$ has zero long run mean and unit local volatility.

The bond price can be calculated as follows

$$P(\tau)(t) = E_t \left[ M_0(T) \left(1 + \gamma e^{-\beta x(T)}\right) \right]$$

$$= M_0(t) E_t \left[ \frac{M_0(T)}{M_0(t)} \right] + \gamma e^{-\beta x(t)} M_0(t) E_t \left[ \frac{\gamma e^{-\beta x(T)} M_0(T)}{\gamma e^{-\beta x(t)} M_0(t)} \right]$$

$$= s(t) P_0^\tau(t) + (1 - s(t)) P_1^\tau(t),$$

where $s(x) = \frac{1}{1 + \gamma e^{-\beta x}}$, and $P_0^\tau(t)$ and $P_1^\tau(t)$ are bond prices in economies with SDFs $M_0(t)$ and $M_1(t) = \gamma e^{-\beta x(t)} M_0(t)$, respectively.

To simplify the expressions we define $\rho_0 = \rho_{0,0}$ and $\lambda_0 = \lambda_{0,0}$. We also impose the following restrictions on the parameters: $\rho_{0,x} = 1$, $\lambda_{0,x} = 0$, $\kappa \beta = 1$, and $\beta = -\lambda_0$. Hence, the bond price simplifies to

$$P^\tau(t) = s(t) e^{A_0^\tau(t) - B_0^\tau(t) x(t)} + (1 - s(t)) e^{-\left(\rho_0 + \frac{1}{2} \lambda_0^2\right) t},$$

where $A_0^\tau(t)$ and $B_0^\tau(t)$ are the deterministic functions of the completely affine Vasicek base model. The expanded model inherits the tractability of the Vasicek base model.
Bond yields are given in closed form and variations in bond yields are driven by a Gaussian state variable which makes it easy to estimate the model.

The short rate, the market price of risk, and the mean and volatility of the instantaneous bond return are nonlinear functions of the state variable $x(t)$. Specifically,

$$ r(t) = \rho_0 + \frac{1}{2} \lambda_0^2 + s(t) \left( x(t) - \frac{1}{2} \lambda_0^2 \right), $$

$$ \Lambda(t) = s(t) \lambda_0, $$

$$ \nu^{(\tau)}(t) = \Lambda(t) - (1 - \omega^{(\tau)}(t)) \left( B_0^*(\tau) + \lambda_0 \right), $$

$$ e^{(\tau)}(t) = \Lambda(t) \Lambda(t) - s(t)(1 - \omega^{(\tau)}(t)) \lambda_0^* \left( B_0^*(\tau) + \lambda_0 \right), $$

where

$$ \omega^{(\tau)}(t) = \frac{1}{1 + \frac{1}{\gamma} e^{A_0^* (\tau)+ (\rho_0 + \frac{1}{2} \lambda_0^2) t + (B_0^*(\tau) + \lambda_0) x(t)}. $$

We can see from equations (25)-(29) that our simple expanded model differs from the completely affine Vasicek base model in three important aspects. First, the volatility of the instantaneous bond return and the instantaneous volatility of yields are time-varying. Second, the market price of risk is a non-linear function of the state variable and hence expected excess returns are moving with the price and the quantity of risk. Third, the short rate $r(t)$ is not spanned by the state variable $x(t)$.

III Estimation

In this section, we estimate a standard and an expanded three-factor essentially affine Gaussian model. The expanded model has the same number of factors and the number of parameter only increases from 23 to 27.

---

6The instantaneous volatility of the bond yield is $-\frac{1}{\gamma} \nu^{(\tau)}(t)$. 

III.A Data

The models are estimated using a monthly panel of zero-coupon Treasury bond yields. We use month-end (continuously compounded) 1-, 2-, 3-, 4-, and 5-years zero-coupon yields extracted from U.S. Treasury security prices by the method of Fama and Bliss (1987). The data is from the Center for Research in Security Prices (CRSP) and covers the period 1952:6 to 2011:12.

III.B Estimation Methodology

The Gaussian model is estimated using the standard Kalman filter. Since there is a non-linear relation between yields and the latent variables in the expanded model, we use the unscented Kalman filter to estimate the expanded model. Christoffersen, Dorion, Jacobs, and Karoui (2012) find that the unscented Kalman filter works well in estimating affine term structure models when highly non-linear instruments are observed. We briefly discuss the setup but refer to Carr and Wu (2009) and Schwartz and Trolle (2012) for a detailed outline of the unscented filter.

We stack the $N$ observed yields in month $t$ in the vector $y_t$, and set the model up in state-space form. The measurement equation is

$$y(t) = f(X(t)) + \epsilon(t), \quad \epsilon(t) \sim N(0, \sigma I_N),$$

(30)

where $f$ is the function determining the relation between the latent variables and yields.\textsuperscript{7} In the Gaussian model this relation is linear and the standard Kalman filter is used while in the expanded model the relation is non-linear and we use the unscented

\textsuperscript{7}$f(X(t)) = -\frac{1}{2} \ln \left( P^{(r)}(X(t)) \right)$ with $P^{(r)}(X(t))$ given in equation (6).
Kalman filter. The transition equation for the latent variables is

\[ X(t + 1) = C + DX(t) + \eta(t + 1), \quad \eta(t) \sim N(0, Q), \]  

(31)

where \( C \) and \( D \) are functions (known in closed-form) entering the one-month ahead expectation of \( X \); i.e. \( E_t(X(t + 1)) = C + DX(t) \). \( Q \) is the variance of \( X(t + 1) \) given \( X(t) \) which is constant since \( X(t) \) is Gaussian.

Since not all of the parameters are identified, we apply the normalizations proposed in Dai and Singleton (2000). Specifically, for the dynamics of \( X \) given in equation (1) we assume that the mean reversion matrix, \( \kappa \), is lower triangular, the mean of the state variables, \( \bar{X} \), is the zero vector, and that the local volatility, \( \sigma_X \), is the identity matrix.

### III.C Estimation Results

We estimate a three-factor Gaussian model and an expanded three-factor Gaussian model with \( \alpha = 1 \) in equation (7).\(^8\) Parameter estimates and log-likelihood values are reported in Table 1. The log-likelihood value of the expanded model is substantially higher than the value of the Gaussian model. Since there are four additional parameters in the expanded model (the scalar \( \gamma \) and the three-dimensional vector \( \beta \)), the likelihood ratio test of \( 2 \times (\text{log-likelihood difference}) \) is \( \chi^2 \)-distributed with 4 degrees of freedom and the 5%, 1%, and 0.1% significance levels are 9.5, 13.3, and 18.5. The likelihood ratio test of the expanded versus the Gaussian model is 258.3, so the expansion is highly significant.\(^9\)

---

\(^8\) We estimated expanded models with different \( \alpha \)'s and found that the model with an \( \alpha \) of 1 had the best overall fit.

\(^9\) Given the significance of the expansion it might be surprising that \( \gamma = 0 \) cannot be rejected, in which case the expanded model collapses to the standard model. The reason is that \( \gamma, \beta, \) and \( \lambda_0 \) are close to being collinear. This suggests that the expansion parameters can be reduced from four to less than four without affecting the properties of the model significantly. We explore this further in Feldhutter, Heyerdahl-Larsen, and Illeditsch (2013).
The bond price in the expanded model is given as \( P^{(\tau)}(t) = s(t)P_0^{(\tau)}(t) + (1 - s(t))P_1^{(\tau)}(t) \) where \( P_0^{(\tau)}(t) \) is the bond price in the base model that is expanded and \( s(t) \) is a weight between 0 and 1. If \( s(t) \) is 1 the bond price in the expanded model collapses to the bond price in the Gaussian base model. Figure 1 shows the weight on the base model. We see that in the 70’s and 80’s the weight is significantly below 1, so the importance of non-linearity becomes particularly important during these periods. The shaded areas in the figure are the NBER recessions and we see that the weight moves away from 1 during recessions.\(^{10}\)

IV  Empirical Results

We focus in this section on expected excess returns, yield volatilities, and the risk-return tradeoff.

IV.A  Expected Excess Returns

Expected excess returns of U.S. Treasury bonds vary over time as documented in Fama (1984), Fama and Bliss (1987) and Campbell and Shiller (1991) (CS). CS document this by regressing future yield changes on the scaled slope of the yield curve. The slope regression coefficient is one if excess holding period returns are constant, but CS find negative regression coefficients.

Panel A in Table 2 shows that both the Gaussian and expanded model replicates the CS regression coefficients; the coefficients are negative and decreasing in maturity consistent with the evidence in the data. It is well-known that the Gaussian model can match this predictability while affine models with stochastic volatility cannot

\(^{10}\)This is consistent with the interpretation of the weight as the surplus consumption ratio in a habit formation model in Appendix B.
(Dai and Singleton (2002), Tang and Xia (2007), and Feldhütter (2008)). Hence, the expansion removes the tension between capturing time variation in excess returns and having stochastic volatility.

Panel B in Table 2 shows the $R^2$'s from regressing realized holding period excess returns on model-implied expected excess returns. Expectations of yields in the Gaussian model are given in closed form and expectations of yields in the expanded Gaussian model are easily calculated using Hermite-Gauss quadrature. For comparison we also include the Cochrane and Piazzesi (2005)-factor because this factor has emerged as a benchmark for predicting bond excess returns. The $R^2$'s for the one-year holding horizon suggests that the Gaussian model does slightly better than the expanded model. Figure 2 plots the realized one-year excess returns along with expected excess returns in the two models. Both return series are averaged over all four bond maturities. The models have similar predictions for excess returns apart from the time of the monetary experiment in the early eighties. The Gaussian model predicts excess returns that are around 2-3% while the expanded model predicts high excess returns often exceeding 5% during that period. It is surprising that the $R^2$'s of the expanded model are lower because the early eighties had two instances where realized returns were more than 10% while this did not occur at any other time in the sample period. Indeed, once we look at excess holding returns beyond one year in Panel B in Table 2 the expanded model outperforms the Gaussian model with $R^2$'s that are close to double to those of the Gaussian model. Interestingly, the expanded model consistently outperforms the Cochrane-Piazzesi single factor at holding horizons beyond one year.

Taken together, the evidence suggests that the expanded model captures a rise in expected excess returns in the early eighties that is missed by the Gaussian model.

\footnote{Almeida, Graveline, and Joslin (2011) show that if options are included in the estimation, affine models with stochastic volatility can match the CS regression coefficients.}
This finding is important not only for term structure modeling but also for the common approach of predicting excess returns in vector-autoregressive (VAR) models (classic examples are Campbell and Ammer (1993) and Ang and Piazzesi (2003)). Joslin, Singleton, and Zhu (2011) show that conditional expected excess returns in a Gaussian model without parameter restrictions are identical to those from an unrestricted VAR model. Thus, any three-factor VAR with yields as factors would miss the rise in expected excess returns.

**IV.B Stochastic Volatilities**

We study in this section to what extent the volatilities of yields (and thus the volatilities of excess returns) in the data can be matched by the expanded model. Yield volatilities in the Gaussian model are constant over time and this is obviously a major shortcoming when studying investment opportunities in the bond market (see Section IV.C) and pricing and risk management of fixed income securities.

We follow the literature and compare model-implied conditional volatilities with an EGARCH(1,1) estimate (see for example Jacobs and Karoui (2009), Almeida, Graveline, and Joslin (2011), and Kim and Singleton (2011)). Specifically, we estimate the conditional volatility of monthly yield changes in the data using the EGARCH(1,1) model and compare it to the model-implied one-month ahead implied volatility (calculated using Gaussian quadrature). The result is shown in Figure 3. There is a clear spike in volatility in the early eighties, and the magnitude of this spike is captured by the expanded model. The average correlation between model-implied and EGARCH(1,1) volatility is 71.5%, similar to the correlations found in Feldhütter (2008) and Jacobs and Karoui (2009) for affine models with one or more CIR processes.

However, a regression of actual yield volatility on model-implied volatility (and
a constant) reveals an important difference between the volatility dynamics in the expanded model and affine models with stochastic volatility. The slope coefficients are 0.87, 0.77, 0.76, 0.77, 0.71 for the 1-, 2-, 3-, 4-, and 5-year bond maturities in the expanded model. The slope coefficients are slightly lower than 1 but they are fairly consistent across maturities. In contrast the slope coefficients of the essentially affine three-factor affine model with one stochastic volatility factor reported in Panel A, Table 5 of Jacobs and Karoui (2009) are 1.92, 1.49, and 1.13 for the 1-, 3-, and 5-year yields. This illustrates the difficulty that affine models with stochastic volatility have in matching volatility; they tend to have too little variation in volatility and this is more pronounced at short maturities. There is a linear relation between yield variance and latent factors in affine models with stochastic volatility and breaking this linearity by our expansion helps capture volatility consistently across maturity.

IV.C Sharpe Ratios

We measure the investment opportunity of a bond at time $t$ with its Sharpe ratio

$$S_t = \frac{E_t(r_{x_t}^{(\tau)})}{\sqrt{Var_t(r_{x_t}^{(\tau)})}},$$

where $r_{x_t}^{(\tau)}$ is one-year excess return on a bond with maturity $\tau$. Gaussian models are benchmark models when examining the time-varying investment opportunities in the bond market.\textsuperscript{12} Viewing the risk-return tradeoff in the bond market through the lens of a Gaussian model can lead to inaccurate conclusions because bond return volatilities in Gaussian models are constant and hence all the variation in the Sharpe ratio must come from the variation in expected excess return. As Duffee (2010a) points out: “existing dynamic term structure models are insufficiently flex-

\textsuperscript{12}Examples include Sangvinatsos and Wachter (2005), Cochrane and Piazzesi (2008), Duffee (2010b), and Joslin, Priebsch, and Singleton (2012).
ible to capture the empirical dynamics of both conditional means and conditional covariances. Thus either the numerator or denominator of the conditional Sharpe ratio is likely misspecified.” The previous sections showed that the expanded model captures expected excess returns at least as well as the Gaussian model and in addition captures the time variation in yield volatilities (and thus the volatilities of excess returns). Therefore, the expanded model provides more realistic estimates of conditional Sharpe ratios.

Figure 4 shows conditional Sharpe ratios in the Gaussian and the expanded model. We calculate for each maturity \( \tau = 2, 3, 4, 5 \) the one-year ahead expected excess return and the square root of the one-year ahead conditional variance of excess return at time \( t \) and take the ratio of the two to determine the conditional Sharpe ratio. We then average across bond maturities.13 Surprisingly, the models agree on Sharpe ratios in the early eighties. Assuming the implications from the expanded model are correct, the Gaussian model underestimates both expected excess returns and volatility of excess returns and gets the ratio roughly right. The difference between the models shows up in the calm periods of the data sample. In these periods the two models agree on expected excess returns as shown in Section IV.A, but the volatility in the Gaussian model is too high because the volatility has to match average sample volatility. As a consequence, the Gaussian model predicts Sharpe ratios that are too low.

13The unconditional Sharpe ratios of the two models are broadly consistent with the average Sharpe ratio in the data. For the full sample, the average Sharpe ratio in the data is 0.23 while it is 0.48 in the Expanded Gaussian model and 0.38 in the Gaussian model. In the first half of the sample the Sharpe ratio is negative -0.14 in the data while it is 0.36 in the Expanded model and 0.26 in the Gaussian model while in the second half the average actual Sharpe ratio is 0.64 while it is 0.59 in the Expanded model and 0.51 in the Gaussian model.
IV.D Linearly Hidden Information

Standard affine term structure models imply that expected excess returns are spanned by yields. However, there is growing evidence that macro variables such as inflation and industrial production growth have predictive power for excess returns and part of this predictive power is orthogonal to the yield curve (Ludvigson and Ng (2009), Cooper and Priestley (2009), Cieslak and Povala (2010), Duffee (2011), Joslin, Priebsch, and Singleton (2012), and Chernov and Mueller (2012)).

This finding motives Gaussian "hidden factor" models where one or more factors determine excess returns but are unrelated to the cross-section of yields. Hidden factors show up either through explicit parameter restrictions (Joslin, Priebsch, and Singleton (2012)) or through filtering in a term structure model with at least five factors (Duffee (2011)).

Our expanded model highlights an alternative channel through which excess returns are imperfectly correlated with the yield curve. In the model expected excess returns are non-linearly related to yields and therefore expected excess returns are not spanned by the cross section of yields. Is the non-linearity strong enough to be empirically relevant? To answer this question we follow Duffee (2011) and calculate the ratio of the variance of expected excess returns projected onto model-implied yields divided by the variance of the true expected excess returns.\textsuperscript{14} In the Gaussian model this ratio is 1. Table 3 shows that in the expanded model this ratio is between 0.71 and 0.86 depending on the forecast horizon and bond maturity. This range compares well with the range 0.53 to 0.70 reported in Duffee (2011)'s five-factor model, which suggests that non-linearity is important in understanding the unspanned part of excess returns.

\textsuperscript{14}We regress expected excess returns on model-implied yields instead of actual yields to highlight the contribution of non-linearity.
Is the linearly hidden part of excess returns related to macro variables? Yes, according to Table 4. We regress expected excess returns on yields and call the residual the linearly hidden part of expected excess returns. We regress the hidden part on inflation and industrial production growth. The table shows a significant negative relation between the linearly hidden part and inflation. This relation occurs because inflation is correlated with the amount of non-linearity in the expanded model, not because inflation has predictive power above what is contained in the yield curve. The amount that inflation and industrial production growth explain of the linearly hidden part ($R^2$ of 6%) is similar to the amount these macro variables explain of Duffee (2011)’s hidden factor ($R^2$ of 8%). This suggests that the importance of non-linearity is similar to the importance of hidden factors in terms of understanding the relation between excess returns and macro variables.

Linearly hidden factors and "truly" hidden factors in the spirit of Duffee (2011) and Joslin, Priebsch, and Singleton (2012) are likely to both play an important rule in understanding expected excess returns. Consistent with this view, Table 3 shows that Duffee’s hidden factor is uncorrelated with the linearly hidden part of excess returns in the expanded model. However, from a modelling perspective the linearly hidden part appears more naturally than the "truly" hidden part. As just shown a significant linearly hidden factor appears in a standard expanded three-factor model. In contrast, Duffee (2011) shows that at least five latent factors along with yield measurement errors are needed to generate a partially hidden factor. Unless economically motivated restrictions are imposed on parameters, a five-factor model leads to overfitting in terms of huge Sharpe ratios (Duffee (2010a)).

15See Table 6 in Duffee (2011)
V Conclusion

In this paper we study the risk and return in the U.S. Treasury bond market. Gaussian models are typically used in studies of the Treasury market, but the models suffer from having constant volatility. We therefore propose a tractable expansion of Gaussian term structure models where the short rate and market prices of risk are non-linear in Gaussian state variables. We estimate a three-factor expanded model and find that the model matches the time variation in both expected excess returns and yield volatilities of U.S. Treasury bonds. Sharpe ratios in the expanded model imply that Treasury bonds are more attractive investments in periods with low volatility compared to the implications of a Gaussian model. A significant part of expected excess returns in the expanded model is not spanned by the cross section of yields. This suggests that more information than previously thought is contained in the yield curve, but in a non-linear way.

Although our empirical analysis has focused on expanding a three-factor essentially affine Gaussian model, our framework can accommodate any term structure model of choice such as affine models with stochastic volatility, quadratic models, and non-linear models. Our expansion introduces new dynamics while keeping the expanded model as tractable as the standard model. Moreover, our framework is not limited to state variables driven by diffusions. The method easily extends to more general processes such as jump-diffusions and continuous time Markov chains.

References

Ait-Sahalia, Yacine, 1996a, Nonparametric pricing of interest rate derivative securities, *Econometrica* 64, 527–560.


Bhamra, Harjoat, and Raman Uppal, 2010, Asset prices with heterogeneity in preferences and beliefs, University of British Columbia.


Cuoco, D., and H. He, 1994, Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets, Wharton School, University of Pennsylvania.


A Proofs

*Proof of Theorem 1.* Using the binomial expansion theorem, the stochastic discount factor in Equation (2) can be expanded as

\[ M(t) = \sum_{n=0}^{\alpha} M_n(t), \tag{33} \]

where

\[ M_n(t) = \left( \frac{\alpha}{n} \right) \gamma^n e^{-n\beta X(t)} M_0(t). \tag{34} \]
The dynamics of the strictly positive stochastic process $M_n(t)$ are

$$\frac{dM_n(t)}{M_n(t)} = -r_n(t) \, dt - \Lambda_n(t)'dW(t),$$

(35)

where

$$\Lambda_n(t) = \Lambda_0(t) + n\Sigma'\beta$$

(36)

$$r_n(t) = r_0(t) + n\beta'\kappa(\bar{X} - X(t)) - \frac{n^2}{2}\beta'\Sigma\Sigma'\beta - n\beta'\Sigma\Lambda_0(t).$$

(37)

Plugging in for $r_0(t)$ and $\Lambda_0(t)$, it is straightforward to show that $\Lambda_n(t)$ and $r_n(t)$ are affine functions of $X(t)$ with coefficients given in Equations (12)-(15). If $M_n(t)$ is interpreted as a stochastic discount factor of an artificial economy index by $n$ then we know that bond prices in this economy belong to the class of essential (exponential) affine Gaussian term structure models and hence

$$P^{(\tau)}(t) = e^{A^*_n(\tau) - B^*_n(\tau)'X(t)},$$

(38)

where coefficient $A^*_n(\tau)$ and the $d$-dimensional vector $B^*_n(\tau)$ solve the ordinary differential equations (10) and (11). Hence, the bond price is

$$P^{(\tau)}(t) = \sum_{n=0}^{\alpha} w_n(t)P^{(\tau)}_n(t),$$

(39)

where $w_n(t)$ is given in equation (9).

\[\square\]

**B  Equilibrium Models**

In this section we show that the functional form of the state price density in equation (2) naturally comes out of several equilibrium models. We need to allow for state
variables that follow arithmetic Brownian motions and hence we rewrite the dynamics of the state vector in equation (1) in the slightly more general form

\[ dX(t) = (\theta - \kappa X(t)) \, dt + \Sigma \, dW(t), \quad (40) \]

where \( \theta \) is \( d \)-dimensional and \( \kappa \) and \( \Sigma \) are \( d \times d \)-dimensional.

In what follows the standard consumption based asset pricing model with a representative agent power utility and log-normally distributed consumption will serve as our benchmark model. Specifically, the state price density takes the following form

\[ M_0(t) = e^{-\rho t} C(t)^{-R}, \quad (41) \]

where \( R \) is the coefficient of RRA and \( C \) is aggregate consumption with dynamics

\[ \frac{dC(t)}{C(t)} = \mu_C dt + \sigma_C' dW(t). \quad (42) \]

The short rate and the market price of risk are both constant and given by

\[ \Lambda_0 = R \sigma_C \]
\[ r_0 = \rho + R \mu_C - \frac{1}{2} R (R + 1) \sigma_C' \sigma_C. \quad (44) \]

Table 5 summarizes the relation between expanded term structure models and the equilibrium models discussed in this section.

**B.A Two Trees**

Cochrane, Longstaff, and Santa-Clara (2008) study an economy in which aggregate consumption is the sum of two Lucas trees. In particular they assume that the
dividends of each tree follow a geometric Brownian motion

\[ dD_i(t) = D_i(t) (\mu_i dt + \sigma_i' dW(t)). \tag{45} \]

Aggregate consumption is \( C(t) = D_1(t) + D_2(t) \). There is a representative agent with power utility and risk aversion \( R \). Hence, the stochastic discount factor is

\[
M(t) = e^{-\rho t} C(t)^{-R} \\
= e^{-\rho t} (D_1(t) + D_2(t))^{-R} \\
= e^{-\rho t} D_1(t)^{-R} \left(1 + \frac{D_2(t)}{D_1(t)} \right)^{-R} \\
= M_0(t) \left(1 + e^{log(D_2(t)) - log(D_1(t))} \right)^{-R}, \tag{46}
\]

where \( M_0(t) = e^{-\rho t} D_1^{-R} \) and \( X(t) = \log(D_1(t)/D_2(t)) \). Equation (46) has the same form as the SDF in equation (2). Specifically, \( \gamma = 1, \beta = 1, \) and \( \alpha = -R \). Note that in this case the state variable is the log-ratio of two geometric Brownian motions and thus \( \kappa = 0 \). The share \( s(X(t)) \) and hence yields are not stationary.

### B.B Multiple Consumption Goods

Models with multiple consumption goods and CES consumption aggregator naturally falls within the functional form of the SDF in equation (2). Consider a setting with two consumption goods. The aggregate output of the two goods are given by

\[ dD_i(t) = D_i(t) (\mu_i dt + \sigma_i' dW(t)). \tag{47} \]

Assume that the representative agent has the following utility over aggregate consumption \( C \),

\[
u(C, t) = e^{-\rho t} \frac{1}{1 - R} C^{1-R}, \tag{48}\]
where
\[ C(C_1, C_2) = \left( \phi^{1-b} C_1^b + (1 - \phi)^{1-b} C_2^b \right)^{\frac{1}{b}}. \] (49)

We use the aggregate consumption bundle as numeraire, and consequently the state price density is
\[ M(t) = e^{-\rho t} C(t)^{-R} \]
\[ = (\phi)^{\frac{b R}{1 - b}} e^{-\rho t} D_1(t)^{-R} \left( 1 + \left( \frac{1 - \phi}{\phi} \right)^{1-b} \left( \frac{D_2(t)}{D_1(t)} \right)^b \right)^{-R}. \] (50)

After normalizing equation (50) has the same form as the SDF in equation (2). Specifically, \( X(t) = \log(D_1(t)/D_2(t)) \), \( \gamma = \left( \frac{1 - \phi}{\phi} \right)^{1-b} \), \( \beta = b \), and \( \alpha = -\frac{R}{b} \). As in the case with Two Trees, the share \( s(X(t)) \) and hence yields are not stationary.

**B.C External Habit Formation**

The utility function in Campbell and Cochrane (1999) is
\[ U(C, H) = e^{-\rho t} \frac{1}{1 - R} (C - H)^{1-R}, \] (51)

where \( H \) is the habit level. Rather than working directly with the habit level, Campbell and Cochrane (1999) define the surplus consumption ratio \( s = \frac{C-H}{C} \). The stochastic discount factor is
\[ M(t) = e^{-\rho t} C(t)^{-R} s(t)^{-R} \]
\[ = M_0(t) s(t)^{-R}. \] (53)
Define the state variable

\[ dX(t) = \kappa (\bar{X} - X(t)) \, dt + bdW(t), \] (54)

where \( \kappa > 0, \sigma > 0 \) and \( b > 0 \). Now let \( s(t) = \frac{1}{1+e^{-\beta X(t)}} \). Note that \( s(t) \) is between 0 and 1. In particular, \( s(t) \) follows

\[ ds(t) = s(t) \left( \mu_s(t) dt + \sigma_s(t) dW(t) \right), \] (55)

where

\[ \mu_s(t) = (1 - s(t)) \left( \beta \kappa (\bar{X} - X(t)) + \frac{1}{2} (1 - 2s(t)) \beta^2 b^2 \right) \] (56)

\[ \sigma_s(t) = (1 - s(t)) \beta b. \] (57)

The functional form of the surplus consumption ratio differs from Campbell and Cochrane (1999). However, note that the surplus consumption ratio is locally perfectly correlated with consumption shocks, mean-reverting and bounded between 0 and 1 just as in Campbell and Cochrane (1999). The state price density can be written as

\[ M(t) = M_0(t) \left( 1 + e^{-\beta X(t)} \right)^R. \] (58)

The above state price density has the same form as equation (2) with parameters \( \gamma = 1, \beta = \beta, \) and \( \alpha = R. \) Note that the state variable \( X \) in this case is mean-reverting and therefore the share \( s(X(t)) \) and hence yields are stationary.
B.D Heterogeneous Beliefs

Consider an economy with two agents that have different beliefs. Let both agents have power utility with the same coefficient of relative risk aversion, $R$. Moreover, assume that aggregate consumption follows the dynamics in equation (42). The agents do not observe the expected growth rate and agree to disagree.\(^{16}\) The equilibrium can be solved by forming the central planner problem with stochastic weight $\lambda$ that captures the agents’ initial relative wealth and their differences in beliefs (see Cuoco and He (1994), Basak and Cuoco (1998) and Basak (2000), for example),

$$
U(C, \lambda) = \max_{\{C_1 + C_2 = C\}} \left( \frac{1}{1 - R} C_1^{1-R} + \lambda \frac{1}{1 - R} C_2^{1-R} \right). \quad (59)
$$

Solving the above problem leads to the optimal consumption of the agents

$$
C_1(t) = s(t) C(t), \quad (60)
$$

$$
C_2(t) = (1 - s(t)) C(t), \quad (61)
$$

where $s(t) = \frac{1}{1 + \lambda(t) \pi}$ is the consumption share of the first agent and $C$ is the aggregate consumption. The state price density as perceived by the first agent is

$$
M(t) = e^{-\rho t} C_1(t)^{-R}
$$

$$
= e^{-\rho t} C(t)^{-R} s(t)^{-R}
$$

$$
= M_0(t) \left( 1 + e^{\frac{1}{R} \log(\lambda(t))} \right)^R. \quad (62)
$$

This has the same form as equation (2) with $X(t) = \log(\lambda(t))$, $\gamma = 1$, $\beta = -\frac{1}{R}$, and $\alpha = R$. The dynamics of the state variable is driven by the log-likelihood ratio of the

---

\(^{16}\)The model can easily be generalised to a setting with disagreement about multiple stochastic processes and learning. For instance, Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2013) show that in a model with disagreement about inflation, the bond prices are weighted averages of quadratic Gaussian term structure models.

32
two agents and consequently the share \( s(X(t)) \) and hence yields are not stationary.

### B.E HARA Utility

Consider a pure exchange economy with a representative agent with utility

\[
u(t, c) = e^{-\rho t} \left( \frac{1}{1-R} \right)^{1-R} (C + b)^{1-R},\]

where \( R > 0 \) and \( b > 0 \). We can write the SDF as

\[
M(t) = e^{-\rho t} C(t)^{-R} \\
= e^{-\rho t} (C(t) + b)^{-R} \\
= e^{-\rho t} C(t)^{-R} \left( 1 + \frac{b}{C(t)} \right)^{-R} \\
= M_0(t) \left( 1 + e^{\log(b) - \log(C(t))} \right)^{-R} \tag{63}
\]

After normalizing equation (63) has the same form as the SDF in equation (2), with state variable \( X(t) = \log(b/C(t)) \), and parameters \( \gamma = 1, \beta = 1 \), and \( \alpha = -R \). Similarly to the model with Two Trees and multiple consumption goods, the share \( s(X(t)) \) and hence yields are non-stationary as the ratio \( b/C(t) \) will eventually converge to zero or infinity depending on the expected growth in the economy.
<table>
<thead>
<tr>
<th></th>
<th>Three-factor Gaussian</th>
<th>Three-factor expanded Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa )</td>
<td>0.645</td>
<td>0.772</td>
</tr>
<tr>
<td></td>
<td>(0.341)</td>
<td>(0.326)</td>
</tr>
<tr>
<td></td>
<td>0.305</td>
<td>0.466</td>
</tr>
<tr>
<td></td>
<td>(0.207)</td>
<td>(0.369)</td>
</tr>
<tr>
<td></td>
<td>0.448</td>
<td>1.02</td>
</tr>
<tr>
<td></td>
<td>(0.197)</td>
<td>(0.305)</td>
</tr>
<tr>
<td>( \rho_0 )</td>
<td>0.00577</td>
<td>-0.00139</td>
</tr>
<tr>
<td></td>
<td>(0.0873)</td>
<td>(0.0522)</td>
</tr>
<tr>
<td>( \rho_X )</td>
<td>0.0019</td>
<td>0.0038</td>
</tr>
<tr>
<td></td>
<td>(0.00873)</td>
<td>(0.00176)</td>
</tr>
<tr>
<td>( \lambda_0 )</td>
<td>0.86</td>
<td>0.356</td>
</tr>
<tr>
<td></td>
<td>(0.757)</td>
<td>(0.166)</td>
</tr>
<tr>
<td>( \lambda_X )</td>
<td>-0.333</td>
<td>-0.622</td>
</tr>
<tr>
<td></td>
<td>(0.277)</td>
<td>(0.222)</td>
</tr>
<tr>
<td></td>
<td>-0.000635</td>
<td>0.000636</td>
</tr>
<tr>
<td></td>
<td>(0.275)</td>
<td>(0.329)</td>
</tr>
<tr>
<td></td>
<td>0.34</td>
<td>0.512</td>
</tr>
<tr>
<td></td>
<td>(0.201)</td>
<td>(0.271)</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0</td>
<td>0.000295</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.00123)</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0</td>
<td>-0.866</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0684)</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>4.63e−007</td>
<td>4.57e−007</td>
</tr>
<tr>
<td></td>
<td>(9.51e−009)</td>
<td>(8.24e−009)</td>
</tr>
<tr>
<td>logL</td>
<td>19578.6</td>
<td>19836.9</td>
</tr>
</tbody>
</table>

Table 1: Parameter Estimates. A standard three-factor Gaussian and an expanded three-factor Gaussian term structure model are estimated using the unscented Kalman filter. This table contains the Kalman filter estimates and asymptotic standard errors (in parenthesis). The models are fitted to monthly Fama-Bliss data of one through five-year zero coupon bond yields from 1952:6 to 2011:12.
### Panel A: Campbell-Shiller regression coefficients

<table>
<thead>
<tr>
<th>bond maturity</th>
<th>2-year</th>
<th>3-year</th>
<th>4-year</th>
<th>5-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual regression coefficient</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.509</td>
<td>-0.849</td>
<td>-1.26</td>
<td>-1.3</td>
<td></td>
</tr>
<tr>
<td>Standard error</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.532</td>
<td>0.62</td>
<td>0.659</td>
<td>0.704</td>
<td></td>
</tr>
<tr>
<td>Gaussian three-factor</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Analytic</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.104</td>
<td>-0.23</td>
<td>-0.408</td>
<td>-0.609</td>
<td></td>
</tr>
<tr>
<td>Simulated</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0112</td>
<td>-0.167</td>
<td>-0.351</td>
<td>-0.547</td>
<td></td>
</tr>
<tr>
<td>Confidence band</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-0.975;1.03)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-1.25;1.06)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-1.63;1.06)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-1.98;1.12)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expanded Gaussian three-factor</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Analytic</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.259</td>
<td>-0.53</td>
<td>-0.745</td>
<td>-0.917</td>
<td></td>
</tr>
<tr>
<td>Simulated</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0335</td>
<td>-0.233</td>
<td>-0.428</td>
<td>-0.58</td>
<td></td>
</tr>
<tr>
<td>Confidence band</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-0.789;1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-1.2;0.95)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-1.52;0.881)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-1.7;0.825)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Panel B: $R^2$ in percent from regressing realized excess returns on expected excess returns

<table>
<thead>
<tr>
<th>bond maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-year holding horizon</td>
</tr>
<tr>
<td>Gaussian three-factor</td>
</tr>
<tr>
<td>14.4</td>
</tr>
<tr>
<td>Expanded Gaussian three-factor</td>
</tr>
<tr>
<td>12.9</td>
</tr>
<tr>
<td>CP single factor</td>
</tr>
<tr>
<td>15.8</td>
</tr>
<tr>
<td>2-year holding horizon</td>
</tr>
<tr>
<td>Gaussian three-factor</td>
</tr>
<tr>
<td>8.3</td>
</tr>
<tr>
<td>Expanded Gaussian three-factor</td>
</tr>
<tr>
<td>13.0</td>
</tr>
<tr>
<td>CP single factor</td>
</tr>
<tr>
<td>10.2</td>
</tr>
<tr>
<td>3-year holding horizon</td>
</tr>
<tr>
<td>Gaussian three-factor</td>
</tr>
<tr>
<td>4.3</td>
</tr>
<tr>
<td>Expanded Gaussian three-factor</td>
</tr>
<tr>
<td>9.7</td>
</tr>
<tr>
<td>CP single factor</td>
</tr>
<tr>
<td>6.4</td>
</tr>
<tr>
<td>4-year holding horizon</td>
</tr>
<tr>
<td>Gaussian three-factor</td>
</tr>
<tr>
<td>3.4</td>
</tr>
<tr>
<td>Expanded Gaussian three-factor</td>
</tr>
<tr>
<td>8.8</td>
</tr>
<tr>
<td>CP single factor</td>
</tr>
<tr>
<td>2.5</td>
</tr>
</tbody>
</table>

**Table 2: Excess return regressions.** Panel A shows the coefficients from the regressions $Y(t+1,n-1) - Y(t,n) = const + \phi^n[Y(t,n) - Y(t,1)] + \text{residual}$, where $Y(t,n)$ is the zero-coupon yield at time $t$ of a bond maturing at time $n$ ($n$ and $t$ are measured in years). The actual coefficients are calculated using monthly Fama-Bliss data of one through five-year zero coupon bond yields from 1952:6 to 2011:12 and standard errors are Hansen and Hodrick (1980) standard errors with 12 lags. For each model the analytical coefficient is based on one simulated sample path of 1,000,000 months, while the simulated coefficient is based on the mean coefficient from 500 simulated paths with length equal to the data. The confidence band is a 95% band based on the 500 simulated paths. Panel B contains the $R^2$ from a regression of realized (log) bond excess returns on expected (log) excess returns in sample. The CP single factor is the Cochrane-Piazzesi factor derived by regressing average (across bonds with maturity 2, 3, 4, and 5 years) 1-year realized excess returns on five forward rates. The excess returns are calculated using Fama-Bliss data of one through five-year zero coupon bond yields from 1952:6 to 2011:12.
<table>
<thead>
<tr>
<th>Bond maturity</th>
<th>Expectation</th>
<th>Expectation projected on yields</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Instantaneous excess returns</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian three-factor model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.99</td>
<td>0.99</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1.95</td>
<td>1.95</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3.27</td>
<td>3.27</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>5.01</td>
<td>5.01</td>
<td>1</td>
</tr>
<tr>
<td>Expanded Gaussian three-factor model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.14</td>
<td>1.70</td>
<td>0.79</td>
</tr>
<tr>
<td>3</td>
<td>3.74</td>
<td>2.88</td>
<td>0.77</td>
</tr>
<tr>
<td>4</td>
<td>5.40</td>
<td>4.02</td>
<td>0.74</td>
</tr>
<tr>
<td>5</td>
<td>7.04</td>
<td>5.04</td>
<td>0.71</td>
</tr>
<tr>
<td><strong>Panel B: Yearly excess returns</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian three-factor model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.14</td>
<td>0.14</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0.47</td>
<td>0.47</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1.02</td>
<td>1.02</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1.83</td>
<td>1.83</td>
<td>1</td>
</tr>
<tr>
<td>Expanded Gaussian three-factor model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.36</td>
<td>0.31</td>
<td>0.86</td>
</tr>
<tr>
<td>3</td>
<td>1.03</td>
<td>0.86</td>
<td>0.84</td>
</tr>
<tr>
<td>4</td>
<td>1.80</td>
<td>1.48</td>
<td>0.82</td>
</tr>
<tr>
<td>5</td>
<td>2.68</td>
<td>2.14</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Table 3: Model-implied population properties of excess returns. This table reports population properties of excess instantaneous and annual log returns of a $n$-year bond. Excess returns are calculated by subtracting the short rate and the yield on one-year bonds, respectively. Conditional expectations are calculated in the model and compared to the conditional expectations derived by linearly projecting the model-implied expectations onto the five model-implied yields. In the Gaussian model (and any other affine model) the former and the latter are the same. Variances are in percent squared.
Table 4: *Projections of the linearly hidden risk premium component in the three-factor expanded Gaussian model.* There is a non-linear relation between expected excess returns and yields in the expanded Gaussian model. We regress average model-implied one-year expected excess returns across the 2-, 3-, 4-, and 5-year bond on model-implied yields. The residual from the regression is the linearly hidden part of the average expected excess returns. We then regress the hidden part on the CPI inflation and industrial production growth over the next 12 months. The inflation and industrial production is the log change over the next 12 months. The Duffee(2011) hidden factor is calculated by downloading the smoothed risk premium factor from Greg Duffee’s webpage and taking the residual from the projection onto the 1- to 5-year yields. The data sample when the hidden factor is included is 1964:1-2007:12. The t-statistics in parentheses are based on Hansen and Hodrick (1980) standard errors with 12 lags.

<table>
<thead>
<tr>
<th>Inflation</th>
<th>Industrial production growth</th>
<th>Duffee(2011) hidden factor</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.04$</td>
<td>$-0.00$</td>
<td>0.34</td>
<td>0.06</td>
</tr>
<tr>
<td>($-2.26$)</td>
<td>($-0.07$)</td>
<td>(0.55)</td>
<td></td>
</tr>
<tr>
<td>$-0.04$</td>
<td>0.00</td>
<td></td>
<td>0.06</td>
</tr>
<tr>
<td>($-2.26$)</td>
<td>(0.10)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.03$</td>
<td>$-0.00$</td>
<td>0.47</td>
<td>0.05</td>
</tr>
<tr>
<td>($-1.63$)</td>
<td>($-0.12$)</td>
<td>(0.79)</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: *Equilibrium Models.* The table shows various equilibrium models and how they map into expanded term structure models.
Figure 1: *The weight on the base model in the expanded model.* The estimated expanded model has bond prices given by $P^{(\tau)}(t) = s(t)P_0^{(\tau)}(t) + (1 - s(t))P_1^{(\tau)}(t)$ where $P_0^{(\tau)}(t)$ is the bond price in the base model that is expanded. $s(t)$ is a weight between 0 and 1 and the figure plots the weight. The shaded areas show NBER recessions.
Figure 2: *Realized and expected excess returns.* This plot shows the average 1-year realized and expected excess return across bonds with a maturity of 2, 3, 4, and 5 year. The ex-post return line is shifted to the left one year so that the realization lines up with the forecast. The data are monthly Fama-Bliss yields of one through five-year zero coupon bond yields from 1952:6 to 2011:12.
Figure 3: Model-implied and $\text{EGARCH}(1,1)$ volatilities of yields. These plots show the model-implied conditional one-month ahead volatility along with an $\text{EGARCH}(1,1)$ estimate of monthly conditional volatility. 'Gaussian' refers to the estimated three-factor Gaussian model while 'Expanded Gaussian' refers to the estimated three-factor expanded Gaussian model. The models are fitted to monthly Fama-Bliss data of one through five-year zero coupon bond yields from 1952:6 to 2011:12.
Figure 4: Conditional Sharpe ratios. The conditional Sharpe ratio for a bond is calculated as $E_t(r_{x,t+12}^{(\tau)})/\sqrt{Var_t(r_{x,t+12}^{(\tau)})}$ where $r_{x,t+12}^{(\tau)}$ is the one-year excess return on a bond with maturity $\tau$. The figure shows the average conditional Sharpe ratio across the 2-, 3-, 4- and 5-year bonds.